axs.1 Introduction

nml:axs:int:sec We have a semantics for the basic modal language in terms of modal models, and a notion of a formula being valid—true at all worlds in all models—or valid with respect to some class of models or frames—true at all worlds in all models in the class, or based on the frame. Logic usually connects such semantic characterizations of validity with a proof-theoretic notion of derivability. The aim is to define a notion of derivability in some system such that a formula is derivable iff it is valid.

The simplest and historically oldest derivation systems are so-called Hilbert-type or axiomatic derivation systems. Hilbert-type derivation systems for many modal logics are relatively easy to construct: they are simple as objects of metatheoretical study (e.g., to prove soundness and completeness). However, they are much harder to use to prove formulas in than, say, natural deduction systems.

In Hilbert-type derivation systems, a derivation of a formula is a sequence of formulas leading from certain axioms, via a handful of inference rules, to the formula in question. Since we want the derivation system to match the semantics, we have to guarantee that the set of derivable formulas are true in all models (or true in all models in which all axioms are true). We'll first isolate some properties of modal logics that are necessary for this to work: the "normal" modal logics. For normal modal logics, there are only two inference rules that need to be assumed: modus ponens and necessitation. As axioms we take all (substitution instances) of tautologies, and, depending on the modal logic we deal with, a number of modal axioms. Even if we are just interested in the class of all models, we must also count all substitution instances of K and Dual as axioms. This alone generates the minimal normal modal logic **K**.

Definition axs.1. The rule of *modus ponens* is the inference schema

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\ \mathrm{MP}$$

We say a formula ψ *follows from* formulas φ, χ by modus ponens iff $\chi \equiv \varphi \rightarrow \psi$.

Definition axs.2. The rule of *necessitation* is the inference schema

$$\frac{\varphi}{\Box\varphi}\ \mathrm{NEC}$$

We say the formula ψ follows from the formulas φ by necessitation iff $\psi \equiv \Box\varphi$.

Definition axs.3. A *derivation* from a set of axioms Σ is a sequence of formulas ψ_1, ψ_2, …, ψ_n, where each ψ_i is either

1. a substitution instance of a tautology, or
2. a substitution instance of a formula in Σ, or
3. follows from two formulas ψ_j, ψ_k with j, $k < i$ by modus ponens, or

4. follows from a formula ψ_j with $j < i$ by necessitation.

If there is such a derivation with $\psi_n \equiv \varphi$, we say that φ is *derivable from* Σ, in symbols $\Sigma \vdash \varphi$.

With this definition, it will turn out that the set of derivable formulas forms a normal modal logic, and that any derivable formula is true in every model in which every axiom is true. This property of derivations is called *soundness*. The converse, *completeness*, is harder to prove.

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Bibliography