

axs.1 Introduction

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We have a semantics for the basic modal language in terms of modal models, and a notion of a **formula** being valid—true at all worlds in all models—or valid with respect to some class of models or frames—true at all worlds in all models in the class, or based on the frame. Logic usually connects such semantic characterizations of validity with a proof-theoretic notion of **derivability**. The aim is to define a notion of **derivability** in some system such that a **formula** is **derivable** iff it is valid.

The simplest and historically oldest **derivation** systems are so-called Hilbert-type or axiomatic **derivation** systems. Hilbert-type **derivation** systems for many modal logics are relatively easy to construct: they are simple as objects of metatheoretical study (e.g., to prove soundness and completeness). However, they are much harder to use to prove **formulas** in than, say, natural deduction systems.

In Hilbert-type **derivation** systems, a derivation of a **formula** is a sequence of **formulas** leading from certain axioms, via a handful of inference rules, to the **formula** in question. Since we want the **derivation** system to match the semantics, we have to guarantee that the set of **derivable** formulas are true in all models (or true in all models in which all axioms are true). We'll first isolate some properties of modal logics that are necessary for this to work: the “normal” modal logics. For normal modal logics, there are only two inference rules that need to be assumed: modus ponens and necessitation. As axioms we take all (substitution instances) of tautologies, and, depending on the modal logic we deal with, a number of modal axioms. Even if we are just interested in the class of all models, we must also count all substitution instances of K and Dual as axioms. This alone generates the minimal normal modal logic **K**.

Definition axs.1. The rule of *modus ponens* is the inference schema

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \text{MP}$$

We say a **formula** ψ follows from **formulas** φ, χ by modus ponens iff $\chi \equiv \varphi \rightarrow \psi$.

Definition axs.2. The rule of *necessitation* is the inference schema

$$\frac{\varphi}{\Box\varphi} \text{NEC}$$

We say the **formula** ψ follows from the **formulas** φ by necessitation iff $\psi \equiv \Box\varphi$.

Definition axs.3. A *derivation* from a set of axioms Σ is a sequence of **formulas** $\psi_1, \psi_2, \dots, \psi_n$, where each ψ_i is either

1. a substitution instance of a tautology, or
2. a substitution instance of a **formula** in Σ , or
3. follows from two **formulas** ψ_j, ψ_k with $j, k < i$ by modus ponens, or

4. follows from a formula ψ_j with $j < i$ by necessitation.

If there is such a derivation with $\psi_n \equiv \varphi$, we say that φ is *derivable from Σ* , in symbols $\Sigma \vdash \varphi$.

With this definition, it will turn out that the set of derivable formulas forms a normal modal logic, and that any derivable formula is true in every model in which every axiom is true. This property of derivations is called *soundness*. The converse, *completeness*, is harder to prove.

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Bibliography