

prf.1 Derived Rules

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Finding and writing **derivations** is obviously difficult, cumbersome, and repetitive. For instance, very often we want to pass from $\varphi \rightarrow \psi$ to $\Box\varphi \rightarrow \Box\psi$, i.e., apply rule RK. That requires an application of NEC, then recording the proper instance of K, then applying MP. Passing from $\varphi \rightarrow \psi$ and $\psi \rightarrow \chi$ to $\varphi \rightarrow \chi$ requires recording the (long) tautological instance

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

and applying MP twice. Often we want to replace a sub-**formula** by a formula we know to be equivalent, e.g., $\Diamond\varphi$ by $\neg\Box\neg\varphi$, or $\neg\neg\varphi$ by φ . So rather than write out the actual **derivation**, it is more convenient to simply record why the intermediate steps are **derivable**. For this purpose, let us collect some facts about **derivability**.

Proposition prf.1. *If $\mathbf{K} \vdash \varphi_1, \dots, \mathbf{K} \vdash \varphi_n$, and ψ follows from $\varphi_1, \dots, \varphi_n$ by propositional logic, then $\mathbf{K} \vdash \psi$.*

Proof. If ψ follows from $\varphi_1, \dots, \varphi_n$ by propositional logic, then

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_n \rightarrow \psi) \dots)$$

is a tautological instance. Applying MP n times gives a **derivation** of ψ . \square

We will indicate use of this proposition by PL.

Proposition prf.2. *If $\mathbf{K} \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_{n-1} \rightarrow \varphi_n) \dots)$ then $\mathbf{K} \vdash \Box\varphi_1 \rightarrow (\Box\varphi_2 \rightarrow \dots (\Box\varphi_{n-1} \rightarrow \Box\varphi_n) \dots)$.*

Proof. By induction on n , just as in the proof of ?? \square

We will indicate use of this proposition by RK. Let's illustrate how these results help establishing **derivability** results more easily.

Proposition prf.3. $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

Proof.

1. $\mathbf{K} \vdash \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$ TAUT
2. $\mathbf{K} \vdash \Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))$ RK, 1
3. $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ PL, 2

\square

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Proposition prf.4. *If $\mathbf{K} \vdash \varphi \leftrightarrow \psi$ and $\mathbf{K} \vdash \chi[\varphi/p]$ then $\mathbf{K} \vdash \chi[B/p]$*

Proof. Exercise. \square

Problem prf.1. Prove **Proposition prf.4** by proving, by induction on the complexity of χ , that if $\mathbf{K} \vdash \varphi \leftrightarrow \psi$ then $\mathbf{K} \vdash \chi[\varphi/p] \leftrightarrow \chi[\psi/p]$.

This proposition comes in handy especially when we want to convert \diamond into \Box (or vice versa), or remove double negations inside a formula. For instance:

Proposition prf.5. $\mathbf{K} \vdash \neg\Box p \rightarrow \diamond\neg p$

Proof.

1. $\mathbf{K} \vdash \diamond\neg p \leftrightarrow \neg\Box\neg\neg p$ DUAL
2. $\mathbf{K} \vdash \neg\Box\neg\neg p \rightarrow \diamond\neg p$ PL, 1
3. $\mathbf{K} \vdash \neg\Box p \rightarrow \diamond\neg p$ re-write p for $\neg\neg p$

□

The following proposition justifies that we can establish derivability results schematically. E.g., the previous proposition does not just establish that $\mathbf{K} \vdash \neg\Box p \rightarrow \diamond\neg p$, but $\mathbf{K} \vdash \neg\Box\varphi \rightarrow \diamond\neg\varphi$ for arbitrary φ .

Proposition prf.6. *If φ is a substitution instance of ψ and $\mathbf{K} \vdash \psi$, then $\mathbf{K} \vdash \varphi$.*

Proof. It is tedious but routine to verify (by induction on the length of the derivation of ψ) that applying a substitution to an entire derivation also results in a correct derivation. Specifically, substitution instances of tautological instances are themselves tautological instances, substitution instances of instances of DUAL and K are themselves instances of DUAL and K, and applications of MP and NEC remain correct when substituting formulas for propositional variables in both premise(s) and conclusion. □

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Bibliography