

## prf.1 Derived Rules

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sec

Finding and writing **derivations** is obviously difficult, cumbersome, and repetitive. For instance, very often we want to pass from  $\varphi \rightarrow \psi$  to  $\Box\varphi \rightarrow \Box\psi$ , i.e., apply rule RK. That requires an application of NEC, then recording the proper instance of K, then applying MP. Passing from  $\varphi \rightarrow \psi$  and  $\psi \rightarrow \chi$  to  $\varphi \rightarrow \chi$  requires recording the (long) tautological instance

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

and applying MP twice. Often we want to replace a sub-**formula** by a formula we know to be equivalent, e.g.,  $\Diamond\varphi$  by  $\neg\Box\neg\varphi$ , or  $\neg\neg\varphi$  by  $\varphi$ . So rather than write out the actual **derivation**, it is more convenient to simply record why the intermediate steps are **derivable**. For this purpose, let us collect some facts about **derivability**.

**Proposition prf.1.** *If  $\mathbf{K} \vdash \varphi_1, \dots, \mathbf{K} \vdash \varphi_n$ , and  $\psi$  follows from  $\varphi_1, \dots, \varphi_n$  by propositional logic, then  $\mathbf{K} \vdash \psi$ .*

*Proof.* If  $\psi$  follows from  $\varphi_1, \dots, \varphi_n$  by propositional logic, then

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_n \rightarrow \psi) \dots)$$

is a tautological instance. Applying MP  $n$  times gives a **derivation** of  $\psi$ .  $\square$

We will indicate use of this proposition by PL.

**Proposition prf.2.** *If  $\mathbf{K} \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_{n-1} \rightarrow \varphi_n) \dots)$  then  $\mathbf{K} \vdash \Box\varphi_1 \rightarrow (\Box\varphi_2 \rightarrow \dots (\Box\varphi_{n-1} \rightarrow \Box\varphi_n) \dots)$ .*

*Proof.* By induction on  $n$ , just as in the proof of ??  $\square$

We will indicate use of this proposition by RK. Let's illustrate how these results help establishing **derivability** results more easily.

**Proposition prf.3.**  $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

*Proof.*

1.  $\mathbf{K} \vdash \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$       TAUT
2.  $\mathbf{K} \vdash \Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))$       RK, 1
3.  $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$       PL, 2

$\square$

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**Proposition prf.4.** *If  $\mathbf{K} \vdash \varphi \leftrightarrow \psi$  and  $\mathbf{K} \vdash \chi[\varphi/p]$  then  $\mathbf{K} \vdash \chi[B/p]$*

*Proof.* Exercise.  $\square$

**Problem prf.1.** Prove **Proposition prf.4** by proving, by induction on the complexity of  $\chi$ , that if  $\mathbf{K} \vdash \varphi \leftrightarrow \psi$  then  $\mathbf{K} \vdash \chi[\varphi/p] \leftrightarrow \chi[\psi/p]$ .

This proposition comes in handy especially when we want to convert  $\diamond$  into  $\Box$  (or vice versa), or remove double negations inside a formula. For instance:

**Proposition prf.5.**  $\mathbf{K} \vdash \neg\Box p \rightarrow \diamond\neg p$

*Proof.*

1.  $\mathbf{K} \vdash \diamond\neg p \leftrightarrow \neg\Box\neg\neg p$  DUAL
2.  $\mathbf{K} \vdash \neg\Box\neg\neg p \rightarrow \diamond\neg p$  PL, 1
3.  $\mathbf{K} \vdash \neg\Box p \rightarrow \diamond\neg p$  re-write  $p$  for  $\neg\neg p$

□

The following proposition justifies that we can establish derivability results schematically. E.g., the previous proposition does not just establish that  $\mathbf{K} \vdash \neg\Box p \rightarrow \diamond\neg p$ , but  $\mathbf{K} \vdash \neg\Box\varphi \rightarrow \diamond\neg\varphi$  for arbitrary  $\varphi$ .

**Proposition prf.6.** *If  $\varphi$  is a substitution instance of  $\psi$  and  $\mathbf{K} \vdash \psi$ , then  $\mathbf{K} \vdash \varphi$ .*

*Proof.* It is tedious but routine to verify (by induction on the length of the derivation of  $\psi$ ) that applying a substitution to an entire derivation also results in a correct derivation. Specifically, substitution instances of tautological instances are themselves tautological instances, substitution instances of instances of DUAL and K are themselves instances of DUAL and K, and applications of MP and NEC remain correct when substituting formulas for propositional variables in both premise(s) and conclusion. □

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## Bibliography