Finding and writing derivations is obviously difficult, cumbersome, and repetitive. For instance, very often we want to pass from $\varphi \rightarrow \psi$ to $\Box \varphi \rightarrow \Box \psi$, i.e., apply rule RK. That requires an application of NEC, then recording the proper instance of K, then applying MP. Passing from $\varphi \rightarrow \psi$ and $\psi \rightarrow \chi$ to $\varphi \rightarrow \chi$ requires recording the (long) tautological instance 

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

and applying MP twice. Often we want to replace a sub-formula by a formula we know to be equivalent, e.g., $\Diamond \varphi$ by $\neg \Box \neg \varphi$, or $\neg \neg \varphi$ by $\varphi$. So rather than write out the actual derivation, it is more convenient to simply record why the intermediate steps are derivable. For this purpose, let us collect some facts about derivability.

**Proposition prf.1.** If $K \vdash \varphi_1, \ldots, K \vdash \varphi_n$, and $\psi$ follows from $\varphi_1, \ldots, \varphi_n$ by propositional logic, then $K \vdash \psi$.

**Proof.** If $\psi$ follows from $\varphi_1, \ldots, \varphi_n$ by propositional logic, then

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_n \rightarrow \psi) \cdots)$$

is a tautological instance. Applying MP $n$ times gives a derivation of $\psi$. □

We will indicate use of this proposition by PL.

**Proposition prf.2.** If $K \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_{n-1} \rightarrow \varphi_n) \cdots)$ then $K \vdash \Box \varphi_1 \rightarrow (\Box \varphi_2 \rightarrow \cdots (\Box \varphi_{n-1} \rightarrow \Box \varphi_n) \cdots)$.

**Proof.** By induction on $n$, just as in the proof of ??.

We will indicate use of this proposition by RK. Let’s illustrate how these results help establishing derivability results more easily.

**Proposition prf.3.** $K \vdash (\Box \varphi \land \Box \psi) \rightarrow \Box(\varphi \land \psi)$

**Proof.**

1. $K \vdash \varphi \rightarrow (\psi \rightarrow (\varphi \land \psi))$ TAUT
2. $K \vdash \Box \varphi \rightarrow (\Box \psi \rightarrow \Box(\varphi \land \psi))$ RK, 1
3. $K \vdash (\Box \varphi \land \Box \psi) \rightarrow \Box(\varphi \land \psi)$ PL, 2

□

**Proposition prf.4.** If $K \vdash \varphi \leftrightarrow \psi$ and $K \vdash \chi[\varphi/p]$ then $K \vdash \chi[B/p]$

**Proof.** Exercise. □

**Problem prf.1.** Prove Proposition prf.4 by proving, by induction on the complexity of $\chi$, that if $K \vdash \varphi \leftrightarrow \psi$ then $K \vdash \chi[\varphi/p] \leftrightarrow \chi[\psi/p]$.

\[\text{derived-rules rev: } 074a3f1 \text{ (2018-11-13) by OLP / CC–BY} \]
This proposition comes in handy especially when we want to convert ♦ into □ (or vice versa), or remove double negations inside a formula. For instance:

**Proposition prf.5.** \( K \vdash \neg \Box p \rightarrow \Diamond \neg p \)

*Proof.*

1. \( K \vdash \Diamond \neg p \leftrightarrow \neg \Box \neg p \) **DUAL**
2. \( K \vdash \neg \Box \neg p \rightarrow \Diamond \neg p \) **PL, 1**
3. \( K \vdash \neg \Box p \rightarrow \Diamond \neg p \) re-write \( p \) for \( \neg \neg p \)

The following proposition justifies that we can establish derivability results schematically. E.g., the previous proposition does not just establish that \( K \vdash \neg \Box p \rightarrow \Diamond \neg p \), but \( K \vdash \neg \Box \varphi \rightarrow \Diamond \neg \varphi \) for arbitrary \( \varphi \).

**Proposition prf.6.** If \( \varphi \) is a substitution instance of \( \psi \) and \( K \vdash \psi \), then \( K \vdash \varphi \).

*Proof.* It is tedious but routine to verify (by induction on the length of the derivation of \( \psi \)) that applying a substitution to an entire derivation also results in a correct derivation. Specifically, substitution instances of tautological instances are themselves tautological instances, substitution instances of instances of **DUAL** and \( K \) are themselves instances of **DUAL** and \( K \), and applications of **MP** and **NEC** remain correct when substituting formulas for propositional variables in both premise(s) and conclusion.

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**Bibliography**