Finding and writing derivations is obviously difficult, cumbersome, and repetitive. For instance, very often we want to pass from $\varphi \rightarrow \psi$ to $\Box \varphi \rightarrow \Box \psi$, i.e., apply rule \textit{rk}. That requires an application of \textit{nec}, then recording the proper instance of $K$, then applying \textit{mp}. Passing from $\varphi \rightarrow \psi$ and $\psi \rightarrow \chi$ to $\varphi \rightarrow \chi$ requires recording the (long) tautological instance

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

and applying \textit{mp} twice. Often we want to replace a sub-formula by a formula we know to be equivalent, e.g., $\Diamond \varphi$ by $\neg \Box \neg \varphi$, or $\neg \neg \varphi$ by $\varphi$. So rather than write out the actual derivation, it is more convenient to simply record why the intermediate steps are derivable. For this purpose, let us collect some facts about derivability.

**Proposition prf.1.** If $K \vdash \varphi_1, \ldots, K \vdash \varphi_n$, and $\psi$ follows from $\varphi_1, \ldots, \varphi_n$ by propositional logic, then $K \vdash \psi$.

**Proof.** If $\psi$ follows from $\varphi_1, \ldots, \varphi_n$ by propositional logic, then

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_n \rightarrow \psi) \cdots)$$

is a tautological instance. Applying \textit{mp} $n$ times gives a derivation of $\psi$. \hfill \Box

We will indicate use of this proposition by \textit{pl}.

**Proposition prf.2.** If $K \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_{n-1} \rightarrow \varphi_n) \ldots)$ then $K \vdash \Box \varphi_1 \rightarrow (\Box \varphi_2 \rightarrow \cdots (\Box \varphi_{n-1} \rightarrow \Box \varphi_n) \ldots)$.

**Proof.** By induction on $n$, just as in the proof of ?? \hfill \Box

We will indicate use of this proposition by \textit{rk}. Let’s illustrate how these results help establishing derivability results more easily.

**Proposition prf.3.** $K \vdash (\Box \varphi \land \Box \psi) \rightarrow \Box (\varphi \land \psi)$

**Proof.**

1. $K \vdash \varphi \rightarrow (\psi \rightarrow (\varphi \land \psi))$ \hfill \text{TAUT}
2. $K \vdash \Box \varphi \rightarrow (\Box \psi \rightarrow \Box (\varphi \land \psi))$ \hfill \text{RK, 1}
3. $K \vdash (\Box \varphi \land \Box \psi) \rightarrow \Box (\varphi \land \psi)$ \hfill \text{PL, 2}

**Proposition prf.4.** If $K \vdash \varphi \leftrightarrow \psi$ and $K \vdash \chi[\varphi/p]$ then $K \vdash \chi[B/p]$

**Proof.** Exercise. \hfill \Box

**Problem prf.1.** Prove Proposition prf.4 by proving, by induction on the complexity of $\chi$, that if $K \vdash \varphi \leftrightarrow \psi$ then $K \vdash \chi[\varphi/p] \leftrightarrow \chi[\psi/p]$.
This proposition comes in handy especially when we want to convert \( \Diamond \) into \( \square \) (or vice versa), or remove double negations inside a formula. For instance:

**Proposition prf.5.** \( K \vdash \neg \square p \rightarrow \Diamond \neg p \)

*Proof.*
1. \( K \vdash \Diamond \neg p \leftrightarrow \neg \square \neg p \quad \text{DUAL} \)
2. \( K \vdash \neg \square \neg p \rightarrow \Diamond \neg p \quad \text{PL}, \ 1 \)
3. \( K \vdash \neg \square p \rightarrow \Diamond \neg p \quad \text{re-write } p \text{ for } \neg \neg p \)

The following proposition justifies that we can establish derivability results schematically. E.g., the previous proposition does not just establish that \( K \vdash \neg \square p \rightarrow \Diamond \neg p \), but \( K \vdash \neg \square \varphi \rightarrow \Diamond \neg \varphi \) for arbitrary \( \varphi \).

**Proposition prf.6.** If \( \varphi \) is a substitution instance of \( \psi \) and \( K \vdash \psi \), then \( K \vdash \varphi \).

*Proof.* It is tedious but routine to verify (by induction on the length of the derivation of \( \psi \)) that applying a substitution to an entire derivation also results in a correct derivation. Specifically, substitution instances of tautological instances are themselves tautological instances, substitution instances of \( \text{dual} \) and \( K \) are themselves instances of \( \text{dual} \) and \( K \), and applications of MP and NEC remain correct when substituting formulas for propositional variables in both premise(s) and conclusion.

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**Bibliography**