prf.1 Consistency

nml:prf:con:

Consistency is an important property of sets of formulas. A set of formulas is sec inconsistent if a contradiction, such as \perp , is derivable from it; and otherwise consistent. If a set is inconsistent, its formulas cannot all be true in a model at a world. For the completeness theorem we prove the converse: every consistent set is true at a world in a model, namely in the "canonical model."

Definition prf.1. A set Γ is *consistent* relatively to a system Σ or, as we will say, Σ -consistent, if and only if $\Gamma \nvDash_{\Sigma} \perp$.

So for instance, the set $\{\Box(p \rightarrow q), \Box p, \neg \Box q\}$ is consistent relatively to propositional logic, but not **K**-consistent. Similarly, the set $\{\Diamond p, \Box \Diamond p \rightarrow q, \neg q\}$ is not K5-consistent.

nml:prf:con: **Proposition prf.2.** Let Γ be a set of formulas. Then:

- 1. Γ is Σ -consistent if and only if there is some formula φ such that $\Gamma \nvDash_{\Sigma} \varphi$.
- 2. $\Gamma \vdash_{\Sigma} \varphi$ if and only if $\Gamma \cup \{\neg \varphi\}$ is not Σ -consistent.
- 3. If Γ is Σ -consistent, then for any formula φ , either $\Gamma \cup \{\varphi\}$ is Σ consistent or $\Gamma \cup \{\neg \varphi\}$ is Σ -consistent.

Proof. These facts follow easily using classical propositional logic. We give the argument for (3). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg \varphi\}$ is Σ -consistent. Then by (2), both $\Gamma, \varphi \vdash_{\Sigma} \bot$ and $\Gamma, \neg \varphi \vdash_{\Sigma} \bot$. By the deduction theorem $\Gamma \vdash_{\Sigma} \varphi \to \bot$ and $\Gamma \vdash_{\Sigma} \neg \varphi \to \bot$. But $(\varphi \to \bot) \to ((\neg \varphi \to \Box)) \to ((\neg \varphi \to \Box))$ \perp) $\rightarrow \perp$) is a tautological instance, hence by ????, $\Gamma \vdash_{\Sigma} \perp$.

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Bibliography

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