Consistency

Consistency is an important property of sets of formulas. A set of formulas is inconsistent if a contradiction, such as ⊥, is derivable from it; and otherwise consistent. If a set is inconsistent, its formulas cannot all be true in a model at a world. For the completeness theorem we prove the converse: every consistent set is true at a world in a model, namely in the “canonical model.”

Definition prf.1. A set \( \Gamma \) is consistent relatively to a system \( \Sigma \) or, as we will say, \( \Sigma \)-consistent, if and only if \( \Gamma \nvdash \Sigma \perp \).

So for instance, the set \{\( \Box(p \rightarrow q), \Box p, \neg \Box q \}\} is consistent relatively to propositional logic, but not \(K\)-consistent. Similarly, the set \{\( \Diamond p, \Box \Diamond p \rightarrow q, \neg q \}\} is not \(K5\)-consistent.

Proposition prf.2. Let \( \Gamma \) be a set of formulas. Then:

1. \( \Gamma \) is \( \Sigma \)-consistent if and only if there is some formula \( \varphi \) such that \( \Gamma \nvdash \Sigma \varphi \).
2. \( \Gamma \vdash_\Sigma \varphi \) if and only if \( \Gamma \cup \{\neg \varphi\} \) is not \( \Sigma \)-consistent.
3. If \( \Gamma \) is \( \Sigma \)-consistent, then for any formula \( \varphi \), either \( \Gamma \cup \{\varphi\} \) is \( \Sigma \)-consistent or \( \Gamma \cup \{\neg \varphi\} \) is \( \Sigma \)-consistent.

Proof. These facts follow easily using classical propositional logic. We give the argument for (3). Proceed contrapositively and suppose neither \( \Gamma \cup \{\varphi\} \) nor \( \Gamma \cup \{\neg \varphi\} \) is \( \Sigma \)-consistent. Then by (2), both \( \Gamma, \varphi \vdash \perp \) and \( \Gamma, \neg \varphi \vdash \perp \). By the deduction theorem \( \Gamma \vdash_\Sigma \varphi \rightarrow \perp \) and \( \Gamma \vdash_\Sigma \neg \varphi \rightarrow \perp \). But \((\varphi \rightarrow \perp) \rightarrow ((\neg \varphi \rightarrow \perp) \rightarrow \perp)\) is a tautological instance, hence by \(???\), \( \Gamma \vdash \perp \). \(\square\)

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Bibliography