

## prf.1 Consistency

mod:prf:con:  
sec

**Definition prf.1.** A set  $\Gamma$  is *consistent* relatively to a system  $\Sigma$  or, as we will say,  $\Sigma$ -consistent, if and only if  $\Gamma \not\vdash_{\Sigma} \perp$ .

So for instance, the set  $\{\Box(p \rightarrow q), \Box p, \neg\Box q\}$  is consistent relatively to propositional logic, but not **K**-consistent. Similarly, the set  $\{\Diamond p, \Box\Diamond p \rightarrow q, \neg q\}$  is not **K5**-consistent.

mod:prf:con:  
prop:consistencyfacts

**Proposition prf.2.** *Let  $\Gamma$  be a set of formulas. Then:*

1. *A set  $\Gamma$  is  $\Sigma$ -consistent if and only if there is some formula  $\varphi$  such that  $\Gamma \not\vdash_{\Sigma} \varphi$ .*
2.  *$\Gamma \vdash_{\Sigma} \varphi$  if and only if  $\Gamma \cup \{\neg\varphi\}$  is not  $\Sigma$ -consistent.*
3. *If  $\Gamma$  is  $\Sigma$ -consistent, then for any formula  $\varphi$ , either  $\Gamma \cup \{\varphi\}$  is  $\Sigma$ -consistent or  $\Gamma \cup \{\neg\varphi\}$  is  $\Sigma$ -consistent.*

mod:prf:con:  
prop:consistencyfacts-b  
mod:prf:con:  
prop:consistencyfacts-c

*Proof.* These fact follow easily using classical propositional logic. We give the argument for (c). Proceed contrapositively and suppose neither  $\Gamma \cup \{\varphi\}$  nor  $\Gamma \cup \{\neg\varphi\}$  is  $\Sigma$ -consistent. Then by (b) both  $\Gamma, \varphi \vdash_{\Sigma} \perp$  and  $\Gamma, \neg\varphi \vdash_{\Sigma} \perp$ . By the deduction theorem  $\Gamma \vdash_{\Sigma} \varphi \rightarrow \perp$  and  $\Gamma \vdash_{\Sigma} \neg\varphi \rightarrow \perp$ . But  $(\varphi \rightarrow \perp) \rightarrow ((\neg\varphi \rightarrow \perp) \rightarrow \perp)$  is a tautological instance, hence by **????**,  $\Gamma \vdash_{\Sigma} \perp$ .  $\square$

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## Bibliography