

prf.1 Consistency

mod:prf:con:
sec

Consistency is an important property of sets of **formulas**. A set of **formulas** is inconsistent if a contradiction, such as \perp , is **derivable** from it; and otherwise consistent. If a set is inconsistent, its **formulas** cannot all be true in a model at a world. For the completeness theorem we prove the converse: every consistent set is true at a world in a model, namely in the “canonical model.”

Definition prf.1. A set Γ is *consistent* relatively to a system Σ or, as we will say, Σ -consistent, if and only if $\Gamma \not\vdash_{\Sigma} \perp$.

So for instance, the set $\{\Box(p \rightarrow q), \Box p, \neg \Box q\}$ is consistent relatively to propositional logic, but not **K**-consistent. Similarly, the set $\{\Diamond p, \Box \Diamond p \rightarrow q, \neg q\}$ is not **K5**-consistent.

mod:prf:con:
prop:consistencyfacts

Proposition prf.2. *Let Γ be a set of **formulas**. Then:*

1. *A set Γ is Σ -consistent if and only if there is some **formula** φ such that $\Gamma \not\vdash_{\Sigma} \varphi$.*
2. *$\Gamma \vdash_{\Sigma} \varphi$ if and only if $\Gamma \cup \{\neg \varphi\}$ is not Σ -consistent.*
3. *If Γ is Σ -consistent, then for any **formula** φ , either $\Gamma \cup \{\varphi\}$ is Σ -consistent or $\Gamma \cup \{\neg \varphi\}$ is Σ -consistent.*

mod:prf:con:
prop:consistencyfacts-b
mod:prf:con:
prop:consistencyfacts-c

Proof. These facts follow easily using classical propositional logic. We give the argument for (3). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg \varphi\}$ is Σ -consistent. Then by (2), both $\Gamma, \varphi \vdash_{\Sigma} \perp$ and $\Gamma, \neg \varphi \vdash_{\Sigma} \perp$. By the deduction theorem $\Gamma \vdash_{\Sigma} \varphi \rightarrow \perp$ and $\Gamma \vdash_{\Sigma} \neg \varphi \rightarrow \perp$. But $(\varphi \rightarrow \perp) \rightarrow ((\neg \varphi \rightarrow \perp) \rightarrow \perp)$ is a tautological instance, hence by **????**, $\Gamma \vdash_{\Sigma} \perp$. \square

Photo Credits

Bibliography