Chapter udf

Axioms, Derivations, and Modal Systems

prf.1 Modal Logics

mod:prf:log:
sec

Definition prf.1. A *modal logic* is a set Σ of modal formulas which is closed under *tautological implication* in the following sense: if $\varphi_1, \dots, \varphi_n \in \Sigma$ and $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \varphi) \cdots)$ is a tautological instance, then $\varphi \in \Sigma$.

Proposition prf.2. *Every modal logic is closed under the rule of* Modus Ponens*:*

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}\ \text{MP}$$

Proof. $(\varphi \to \psi) \to (\varphi \to \psi)$ is tautological instance, hence if $\varphi \to \psi$ and φ are in Σ, so is ψ. □

prf.2 Normal Modal Logics

mod:prf:nor:
sec

Definition prf.3. A modal logic Σ is *normal* if it is closed under the rule RK:

$$\frac{\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \cdots)}{\Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n) \cdots)}\ \text{RK}$$

Observe that while tautological implication is "fine-grained" enough to preserve *truth at a world*, the rule RK only preserves *truth in a model* (and hence also validity in a frame or in a class of frames).

Proposition prf.4. *Every normal modal logic Σ is closed under the rule of* Necessitation*:*

$$\frac{\varphi}{\Box\varphi}\ \text{NEC}$$

Proof. NEC is just the special case of RK when $n = 1$. $\Box$

Proposition prf.5. *Every normal modal logic Σ contains every instance of* K.

Proof. In fact, K follows from rule RK: $(\varphi \to \psi) \to (\varphi \to \psi)$ is in Σ since it is a tautological instance; one application of RK gives that $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is in Σ as well. $\Box$

Proposition prf.6. *Every normal modal logic Σ contains $\neg\Diamond\bot$.* mod:prf:nor: prop:notDiamondBot

Problem prf.1. Prove Proposition prf.6.

prf.3 Modal Systems

mod:prf:sys: sec

Proposition prf.7. *Let $\varphi_1, \ldots, \varphi_n$ be schemas. Then there is a smallest modal logic Σ containing all instances of $\varphi_1, \ldots, \varphi_n$. Such a modal logic is called a* modal system *and denoted by $\mathbf{K}\varphi_1 \ldots \varphi_n$. The smallest normal modal logic is denoted by* $\mathbf{K}$.

Proof. Given $\varphi_1, \ldots, \varphi_n$, define Σ as the intersection of all normal modal logics containing all instances of $\varphi_1, \ldots, \varphi_n$. The intersection is non-empty as $\mathrm{Frm}(\mathcal{L})$, the set of all formulas, is such a modal logic. $\Box$

prf.4 Logics Defined by Proofs

mod:prf:prf: sec

Definition prf.8. Given a modal system $\mathbf{K}\varphi_1 \ldots \varphi_n$ and a formula ψ we say that ψ is *derivable* in $\mathbf{K}\varphi_1 \ldots \varphi_n$, written $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$, if and only if there are formulas $\chi_1, \ldots, \chi_k$ such that $\chi_k = \psi$ and each χ_i is either a tautological instance, or an instance of the schemas K, $\varphi_1, \ldots, \varphi_n$, or it follows from previous formulas by means of the rules MP or NEC.

The following proposition allows us to show that $\psi \in \Sigma$ by exhibiting a Σ-proof of ψ.

Proposition prf.9. $\mathbf{K}\varphi_1 \ldots \varphi_n = \{\psi : \mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi\}$.

Proof. We use induction on the length of proofs to show that $\{\psi : \mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi\} \subseteq \mathbf{K}\varphi_1 \ldots \varphi_n$. The converse inclusion follows by showing that $\{\psi : \mathbf{K}\varphi_1 \ldots \varphi_n \vdash \varphi\}$ is a normal modal logic containing all the instances of the schemas $\varphi_1, \ldots, \varphi_n$, and the observation that $\mathbf{K}\varphi_1 \ldots \varphi_n$ is, by definition, the smallest such logic. $\Box$

prf.5 Proofs in K

mod:prf:prk: sec

In order to practice proofs in the smallest modal system, we show the valid formulas on the left-hand side of the table of **??** can all be given **K**-proofs. Justifications for steps that are either tautological instances or follow by tautological implication from previous one are just marked "PL" (for "Propositional Logic").

Proposition prf.10. $\mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$

Proof.

1.	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	PL
2.	$\Box[(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	NEC
3.	$\Box(\varphi \to \psi) \to \Box(\neg\psi \to \neg\varphi)$	K, MP
4.	$\Box(\neg\psi \to \neg\varphi) \to (\Box\neg\psi \to \Box\neg\varphi)$	K
5.	$\Box(\varphi \to \psi) \to (\Box\neg\psi \to \Box\neg\varphi)$	PL, 3,4
6.	$(\Box\neg\psi \to \Box\neg\varphi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$	PL
7.	$\Box(\varphi \to \psi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$	PL, 5, 6
8.	$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	re-writing.

□

Proposition prf.11. $\mathbf{K} \vdash \Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$

Proof.

1.	$\varphi \to (\neg\psi \to \neg(\varphi \to \psi))$	PL
2.	$\Box[\varphi \to (\neg\psi \to \neg(\varphi \to \psi))]$	NEC
3.	$\Box\varphi \to \Box(\neg\psi \to \neg(\varphi \to \psi))$	K
4.	$\Box\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \to \psi))$	K
5.	$\Box\varphi \to (\neg\Box\neg(\varphi \to \psi) \to \neg\Box\neg\psi)$	PL
6.	$\Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$	re-writing.

□

Proposition prf.12. $\mathbf{K} \vdash \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$

Proof.

1.	$\neg(\varphi \to \neg\psi) \to \varphi$	PL
2.	$\Box\neg(\varphi \to \neg\psi) \to \Box\varphi$	NEC, K
3.	$\Box(\varphi \wedge \psi) \to \Box\varphi$	re-writing
4.	$\Box(\varphi \wedge \psi) \to \Box\psi$	similarly
5.	$\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$	PL, 3,4.

□

Proposition prf.13. $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

Proof.

1.	$\varphi \to (\psi \to (\varphi \land \psi))$	PL
2.	$\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi))$	NEC, K
3.	$(\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$	PL, 2

$\square$

Proposition prf.14. $\mathbf{K} \vdash \Box\varphi \to \Box(\psi \to \varphi)$

Proof.

1.	$\varphi \to (\psi \to \varphi)$	PL
2.	$\Box\varphi \to \Box(\psi \to \varphi)$	NEC, K.

$\square$

Proposition prf.15. $\mathbf{K} \vdash \neg\Diamond\varphi \to \Box(\varphi \to \psi)$

Proof.

1.	$\neg\varphi \to (\varphi \to \psi)$	PL
2.	$\Box\neg\varphi \to \Box(\varphi \to \psi)$	NEC, K
3.	$\neg\neg\Box\neg\varphi \to \Box(\varphi \to \psi)$	PL
4.	$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	re-writing.

$\square$

Proposition prf.16. $\mathbf{K} \vdash (\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi)$

Proof.

1.	$\neg(\neg\varphi \to \psi) \to \neg\varphi$	PL
2.	$\neg(\varphi \lor \psi) \to \neg\varphi$	PL, 1
3.	$\Box\neg(\varphi \lor \psi) \to \Box\neg\varphi$	NEC, K
4.	$\neg\Box\neg\varphi \to \neg\Box\neg(\varphi \lor \psi)$	PL
5.	$\Diamond\varphi \to \Diamond(\varphi \lor \psi)$	re-writing
6.	$\Diamond\psi \to \Diamond(\varphi \lor \psi)$	similarly
7.	$(\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi)$	PL, 5, 6.

$\square$

Proposition prf.17. $\mathbf{K} \vdash \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$

Proof.

1.	$\neg\varphi \to (\neg\psi \to \neg(\varphi \lor \psi)$	PL
2.	$\Box\neg\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \lor \psi)$	NEC, K
3.	$\Box\neg\varphi \to (\neg\Box\neg(\varphi \lor \psi) \to \neg\Box\neg\psi))$	PL, 2
4.	$\neg\Box\neg(\varphi \lor \psi) \to (\Box\neg\varphi \to \neg\Box\neg\psi)$	PL
5.	$\neg\Box\neg(\varphi \lor \psi) \to (\neg\neg\Box\neg\psi \to \neg\Box\neg\varphi)$	PL
6.	$\Diamond(\varphi \lor \psi) \to (\neg\Diamond\psi \to \Diamond\varphi)$	re-writing
7.	$\Diamond(\varphi \lor \psi) \to (\Diamond\psi \lor \Diamond\varphi)$	PL.

$\square$

Problem prf.2. Provide **K**-proofs of the following:

1. $\Diamond\neg\bot \to (\Box\varphi \to \Diamond\varphi)$;
2. $\Box(\varphi \lor \psi) \to (\Diamond\varphi \lor \Box\psi)$;
3. $(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$.

prf.6 Dual Schemas

mod:prf:dua: sec

mod:prf:dua: def:duals

Definition prf.18. Each of the schemas T, B, 4, and 5 has a *dual*, denoted by a subscripted diamond, as follows:

$T_\Diamond$: $\varphi \to \Diamond\varphi$
$B_\Diamond$: $\Diamond\Box\varphi \to \varphi$
$4_\Diamond$: $\Diamond\Diamond\varphi \to \Diamond\varphi$
$5_\Diamond$: $\Diamond\Box\varphi \to \Box\varphi$

Each of the dual above schemas is obtained from the corresponding schema by replacing $\neg\varphi$ for φ, contraposing, and re-writing. Schema D is its own dual (modulo the replacement of $\neg\Diamond\neg$ by $\Box$).

Problem prf.3. Show that for each schema φ in Definition prf.18: $\mathbf{K} \vdash \varphi \leftrightarrow \varphi_\Diamond$.

prf.7 Proofs in Modal Systems

mod:prf:prs: sec

We now come to proofs in systems of modal logic other than **K**.

mod:prf:prs: prop:S5facts

Proposition prf.19. *The following provability results obtain:*

1. $\mathbf{KT5} \vdash$ B;
2. $\mathbf{KT5} \vdash 4$;
3. $\mathbf{KDB4} \vdash$ T;
4. $\mathbf{KB4} \vdash 5$;
5. $\mathbf{KB5} \vdash 4$;
6. *mod:prf:prs: prop:S5facts-KT-D* $\mathbf{KT} \vdash$ D.

Proof. We exhibit proofs for each.

1. $\mathbf{KT5} \vdash$ B:

1.	$\Diamond\varphi \to \Box\Diamond\varphi$	5
2.	$\varphi \to \Diamond\varphi$	T$\Diamond$
3.	$\varphi \to \Box\Diamond\varphi$	PL.

2. **KT5** $\vdash$ 4:

1.	$\Diamond\Box\varphi \to \Box\Diamond\Box\varphi$	5 with $\Box\varphi$ for φ
2.	$\Box\varphi \to \Diamond\Box\varphi$	$\mathrm{T}_\Diamond$ with $\Box\varphi$ for φ
3.	$\Box\varphi \to \Box\Diamond\Box\varphi$	PL, 1, 2
4.	$\Diamond\Box\varphi \to \Box\varphi$	$5_\Diamond$
5.	$\Box\Diamond\Box\varphi \to \Box\Box\varphi$	RK, 4
6.	$\Box\varphi \to \Box\Box\varphi$	PL, 3, 5.

3. **KDB4** $\vdash$ T:

1.	$\Diamond\Box\varphi \to \varphi$	$\mathrm{B}_\Diamond$
2.	$\Box\Box\varphi \to \Diamond\Box\varphi$	D with $\Box\varphi$ for φ
3.	$\Box\Box\varphi \to \varphi$	PL1, 2
4.	$\Box\varphi \to \Box\Box\varphi$	4
5.	$\Box\varphi \to \varphi$	PL, 1, 4.

4. **KB4** $\vdash$ 5:

1.	$\Diamond\varphi \to \Box\Diamond\Diamond\varphi$	B with $\Diamond\varphi$ for φ
2.	$\Diamond\Diamond\varphi \to \Diamond\varphi$	$4_\Diamond$
3.	$\Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi$	RK, 2
4.	$\Diamond\varphi \to \Box\Diamond\varphi$	PL, 1, 3.

5. **KB5** $\vdash$ 4:

1.	$\Box\varphi \to \Box\Diamond\Box\varphi$	B with $\Box\varphi$ for φ
2.	$\Diamond\Box\varphi \to \Box\varphi$	$5_\Diamond$
3.	$\Box\Diamond\Box\varphi \to \Box\Box\varphi$	RK, 2
4.	$\Box\varphi \to \Box\Box\varphi$	PL, 1, 3.

6. **KT** $\vdash$ D:

1.	$\Box\varphi \to \varphi$	T
2.	$\varphi \to \Diamond\varphi$	$\mathrm{T}_\Diamond$
3.	$\Box\varphi \to \Diamond\varphi$	PL, 1, 2

$\Box$

Proposition prf.20. **KTB4** = **KT5** = **KDB4** = **KDB5**. *mod:prf:prs: prop:S5*

Problem prf.4. Prove Proposition prf.20.

Definition prf.21. Following tradition, we define **S4** to be the system **KT4**, and **S5** the system **KTB4**.

Proposition prf.20 shows that the classical system **S5** has several equivalent axiomatizations (see **??**).

prf.8 Soundness

mod:prf:snd:sec

mod:prf:snd:thm:soundness **Theorem prf.22** (Soundness Theorem)**.** *If schemas $\varphi_1, \ldots, \varphi_n$ are valid in the classes of models $\mathcal{C}_n, \ldots, \mathcal{C}_n$, respectively, then $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$ implies that ψ is valid in the class of models $\mathcal{C}_n \cap \cdots \cap \mathcal{C}_n$.*

Proof. By induction on length of proofs. For brevity, put $\mathcal{C} = \mathcal{C}_n \cap \cdots \cap \mathcal{C}_n$.

1. Induction Basis: If ψ has a proof of length 1, then it is either a tautological instance or an instance of K, or an instance of one of the schemas. In the first case, ψ is valid in $\mathcal{C}$, since tautological instance are valid in *any* class of models, by **??**. Similarly in the second case, by **??**. Finally in the third case, since ψ is valid in $\mathcal{C}_i$ and $\mathcal{C} \subseteq \mathcal{C}_i$, we have that ψ is valid in $\mathcal{C}$ as well.

2. Inductive step: Suppose ψ has a proof of length $k > 1$. If ψ is a tautological instance or an instance of one of the schemas, we proceed as in the previous step. So suppose ψ is obtained by MP from previous formulas $\chi \to \psi$ and χ. Then $\chi \to \psi$ and χ have proofs of length $< k$, and by inductive hypothesis they are valid in $\mathcal{C}$. By **??**, ψ is valid in $\mathcal{C}$ as well. Finally suppose ψ is obtained by NEC from χ (so that $\psi = \Box\chi$). By inductive hypothesis, χ is valid in $\mathcal{C}$, and by **??** so is φ. □

prf.9 Showing Systems are Distinct

mod:prf:dis:sec In section prf.7 we saw how to prove that two systems of modal logic are in fact the same system. Theorem prf.22 allows us to show that two modal systems Σ and Σ' are distinct, by finding a formula φ such that $\Sigma' \vdash \varphi$ that fails in a model of Σ.

Proposition prf.23. KD $\subsetneq$ KT

Proof. This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show $\mathbf{KD} \subseteq \mathbf{KT}$ we need to see that $\mathbf{KD} \vdash \psi$ implies $\mathbf{KT} \vdash \psi$, which follows from $\mathbf{KT} \vdash \mathbf{D}$, as shown in Proposition prf.19(6). To show that the inclusion is proper, by Soundness (Theorem prf.22), it suffices to exhibit a model of **KD** where some instance $\Box\varphi \to \varphi$ of T fails (an easy task left as an exercise), for then by Soundness $\mathbf{KD} \nvdash \Box\varphi \to \varphi$. □

Proposition prf.24. KB $\neq$ K4.

Proof. We construct a symmetric model where some instance of 4 fails; since obviously the instance is derivable for **K4** but not in **KB**, it will follow $\mathbf{K4} \nsubseteq \mathbf{KB}$. Consider the symmetric model $\mathfrak{M}$ of Figure prf.1 Since the model is symmetric, K and B are true in $\mathfrak{M}$ (by **??** and **??**, respectively). However, $\mathfrak{M}, w \nVdash \Box p \to \Box\Box p$. □

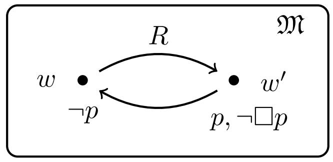

Figure prf.1: A symmetric model falsifying an instance of 4.

mod:prf:dis: fig:Bnot4

Theorem prf.25. **KTB** $\nvdash$ **4** *and* **KTB** $\nvdash$ **5**. *mod:prf:dis: thm:KTBnot45*

Proof. By **??** we know that all instances of T and B are true in each reflexive symmetric model (respectively). So by Soundness it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figure Figure prf.2. Then $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$, so the instance of 4 with $\varphi = p$ fails at w_1. Similarly, $\mathfrak{M}, w_2 \nVdash \Diamond\neg p \to \Box\Diamond\neg p$, so the instance of 5 with $\varphi = \neg p$ fails at w_2. $\Box$

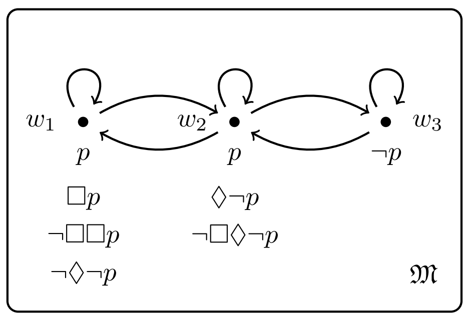

Figure prf.2: The model for Theorem prf.25.

mod:prf:dis: fig:KTBnot45

Theorem prf.26. **KD5** $\neq$ **KT4** $=$ **S4**. *mod:prf:dis: thm:KD5not4*

Proof. By **??** we know that all instances of D and 5 to be true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figure prf.3, and notice that $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$. $\Box$

Problem prf.5. Give an alternative proof of Theorem prf.26 using a model with 3 worlds.

Problem prf.6. Provide a single reflexive transitive model showing that both **KT4** $\nvdash$ B and **KT4** $\nvdash$ 5.

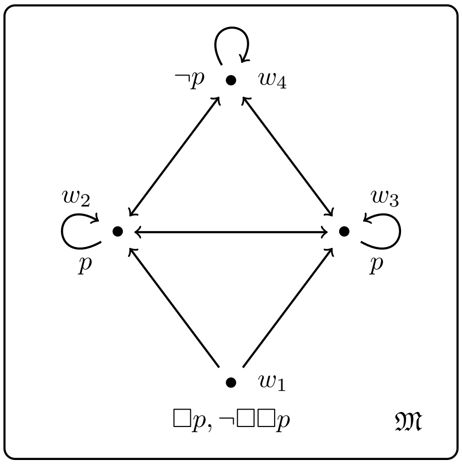

Figure prf.3: The model for Theorem prf.26.

mod:prf:dis: fig:KD5not4

prf.10 Derivability from a Set of Formulas

mod:prf:prg: sec

In section prf.7 we defined a notion of provability of a formula in a system Σ. We now extend this notion to provability in Σ from formulas in a set Γ.

mod:prf:prg: defn:Gammaproves

Definition prf.27. A formula φ is derivable in a system Σ from a set of formulas Γ, written $\Gamma \vdash_\Sigma \varphi$ if and only if there are $\psi_1, \dots, \psi_n \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$.

prf.11 Properties of Derivability

mod:prf:prp: sec

mod:prf:prp: prop:derivabilityfacts

Proposition prf.28. *Let Σ be a modal system and Γ a set of modal formulas. The following properties hold:*

1. Monotony*: If $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_\Sigma \varphi$;*
2. Reflexivity*: If $\varphi \in \Gamma$ then $\Gamma \vdash_\Sigma \varphi$;*
3. Cut*: If $\Gamma \vdash_\Sigma \varphi$ and $\Delta \cup \{\varphi\} \vdash_\Sigma \psi$ then $\Gamma \cup \Delta \vdash_\Sigma \psi$;*
4. Deduction theorem*: $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi$ if and only if $\Gamma \vdash_\Sigma \psi \to \varphi$;*
5. *mod:prf:prp: prop:derivabilityfacts-ruleT* Rule T*: If $\Gamma \vdash_\Sigma \varphi_1$ and … and $\Gamma \vdash_\Sigma \varphi_n$ and $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \psi) \cdots)$ is a tautological instance, then $\Gamma \vdash_\Sigma \psi$.*

The proof is an easy exercise. Part (5) of Proposition prf.28 gives us that, for instance, if $\Gamma \vdash_\Sigma \varphi \lor \psi$ and $\Gamma \vdash_\Sigma \neg\varphi$, then $\Gamma \vdash_\Sigma \psi$. Also, in what follows, we write $\Gamma, \varphi \vdash_\Sigma \psi$ instead of $\Gamma \cup \{\varphi\} \vdash_\Sigma \psi$.

Definition prf.29. A set Γ is *deductively closed* relatively to a system Σ if and only if $\Gamma \vdash_\Sigma \varphi$ implies $\varphi \in \Gamma$.

prf.12 Consistency

mod:prf:con:
sec

Definition prf.30. A set Γ is *consistent* relatively to a system Σ or, as we wil say, Σ-consistent, if and only if $\Gamma \nvdash_\Sigma \bot$.

So for instance, the set $\{\Box(p \to q), \Box p, \neg\Box q\}$ is consistent relatively to propositional logic, but not **K**-consistent. Similarly, the set $\{\Diamond p, \Box\Diamond p \to q, \neg q\}$ is not **K5**-consistent.

Proposition prf.31. *Let Γ be a set of formulas. Then:* *mod:prf:con: prop:consistencyfacts*

1. *A set Γ is Σ-consistent if and only if there is some formula φ such that $\Gamma \nvdash_\Sigma \varphi$.*
2. *$\Gamma \vdash_\Sigma \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is not Σ-consistent.* *mod:prf:con: prop:consistencyfacts-b*
3. *If Γ is Σ-consistent, then for any formula φ, either $\Gamma \cup \{\varphi\}$ is Σ-consistent or $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent.* *mod:prf:con: prop:consistencyfacts-c*

Proof. These fact follow easily using classical propositional logic. We give the argument for (c). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent. Then by (b) both $\Gamma, \varphi \vdash_\Sigma \bot$ and $\Gamma, \neg\varphi \vdash_\Sigma \bot$. By the deduction theorem $\Gamma \vdash_\Sigma \varphi \to \bot$ and $\Gamma \vdash_\Sigma \neg\varphi \to \bot$. But $(\varphi \to \bot) \to ((\neg\varphi \to \bot) \to \bot)$ is a tautological instance, hence by Proposition prf.28(5), $\Gamma \vdash_\Sigma \bot$. □

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Bibliography