Chapter udf

Axiomatic Derivations

axs.1 Introduction

mod:axs:int:sec

We have a semantics for the basic modal language in terms of modal models, and a notion of a formula being valid—true at all worlds in all models—or valid with respect to some class of models or frames—true at all worlds in all models in the class, or based on the frame. Logic usually connects such semantic characterizations of validity with a proof-theoretic notion of derivability. The aim is to define a notion of derivability in some system such that a formula is derivable iff it is valid.

The simplest and historically oldest derivation systems are so-called Hilbert-type or axiomatic derivation systems. Hilbert-type derivation systems for many modal logics are relatively easy to construct: they are simple as objects of metatheoretical study (e.g., to prove soundness and completeness). However, they are much harder to use to prove formulas in than, say, natural deduction systems.

In Hilbert-type derivation systems, a derivation of a formula is a sequence of formulas leading from certain axioms, via a handful of inference rules, to the formula in question. Since we want the derivation system to match the semantics, we have to guarantee that the set of derivable formulas are true in all models (or true in all models in which all axioms are true). We'll first isolate some properties of modal logics that are necessary for this to work: the "normal" modal logics. For normal modal logics, there are only two inference rules that need to be assumed: modus ponens and necessitation. As axioms we take all (substitution instances) of tautologies, and, depending on the modal logic we deal with, a number of modal axioms. Even if we are just interested in the class of all models, we must also count all substitution instances of K and Dual as axioms. This alone generates the minimal normal modal logic **K**.

Definition axs.1. The rule of *modus ponens* is the inference schema

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ \text{MP}$$

We say a formula ψ *follows from* formulas φ, χ by modus ponens iff $\chi \equiv \varphi \to \psi$.

Definition axs.2. The rule of *necessitation* is the inference schema

$$\frac{\varphi}{\Box\varphi}\ \text{NEC}$$

We say the formula ψ follows from the formulas φ by necessitation iff $\psi \equiv \Box\varphi$.

Definition axs.3. A *derivation* from a set of axioms Σ is a sequence of formulas ψ_1, ψ_2, …, ψ_n, where each ψ_i is either

1. a substitution instance of a tautology, or
2. a substitution instance of a formula in Σ, or
3. follows from two formulas ψ_j, ψ_k with j, $k < i$ by modus ponens, or
4. follows from a formula ψ_j with $j < i$ by necessitation.

If there is such a derivation with $\psi_n \equiv \varphi$, we say that φ is *derivable from* Σ, in symbols $\Sigma \vdash \varphi$.

With this definition, it will turn out that the set of derivable formulas forms a normal modal logic, and that any derivable formula is true in every model in which every axiom is true. This property of derivations is called *soundness*. The converse, *completeness*, is harder to prove.

prf.2 Normal Modal Logics

Not every set of modal formulas can easily be characterized as those formulas derivable from a set of axioms. We want modal logics to be well-behaved. First of all, everything we can derive in classical propositional logic should still be derivable, of course taking into account that the formulas may now contain also $\Box$ and $\Diamond$. To this end, we require that a modal logic contain all tautological instances and be closed under modus ponens. mod:prf:nor:sec

Definition prf.4. A *modal logic* is a set Σ of modal formulas which

1. contains all tautologies, and
2. is closed under substitution, i.e., if $\varphi \in \Sigma$, and θ_1, …, θ_n are formulas, then
$$\varphi[\theta_1/p_1, \dots, \theta_n/p_n] \in \Sigma,$$
3. is closed under *modus ponens*, i.e., if φ and $\varphi \to \psi \in \Sigma$, then $\psi \in \Sigma$.

In order to use the relational semantics for modal logics, we also have to require that all formulas valid in all modal models are included. It turns out that this requirement is met as soon as all instances of K and DUAL are derivable, and whenever a formula φ is derivable, so is $\Box\varphi$. A modal logic that satisfies these conditions is called *normal*. (Of course, there are also non-normal modal logics, but the usual relational models are not adequate for them.)

Definition prf.5. A modal logic Σ is *normal* if it contains

$$\Box(p \to q) \to (\Box p \to \Box q), \tag{K}$$
$$\Diamond p \leftrightarrow \neg\Box\neg p \tag{DUAL}$$

and is closed under *necessitation*, i.e., if $\varphi \in \Sigma$, then $\Box\varphi \in \Sigma$.

Observe that while tautological implication is "fine-grained" enough to preserve *truth at a world*, the rule NEC only preserves *truth in a model* (and hence also validity in a frame or in a class of frames).

mod:prf:nor:prop:rk **Proposition prf.6.** *Every normal modal logic is closed under rule* RK*,*

$$\frac{\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \cdots)}{\Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n) \cdots).}\ \text{RK}$$

Proof. By induction on n: If $n = 1$, then the rule is just NEC, and every normal modal logic is closed under NEC.

Now suppose the result holds for $n - 1$; we show it holds for n.

Assume

$$\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \cdots) \in \Sigma$$

By the induction hypothesis, we have

$$\Box\varphi_1 \to (\Box\varphi_2 \to \cdots \Box(\varphi_{n-1} \to \varphi_n) \cdots) \in \Sigma$$

Since Σ is a normal modal logic, it contains all instances of K, in particular

$$\Box(\varphi_{n-1} \to \varphi_n) \to (\Box\varphi_{n-1} \to \Box\varphi_n) \in \Sigma$$

Using modus ponens and suitable tautological instances we get

$$\Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n) \cdots) \in \Sigma.$$ □

mod:prf:nor:prop:notDiamondBot **Proposition prf.7.** *Every normal modal logic Σ contains $\neg\Diamond\bot$.*

Problem prf.1. Prove Proposition prf.7.

Proposition prf.8. *Let $\varphi_1, \ldots, \varphi_n$ be formulas. Then there is a smallest modal logic Σ containing all instances of $\varphi_1, \ldots, \varphi_n$.*

Proof. Given $\varphi_1, \ldots, \varphi_n$, define Σ as the intersection of all normal modal logics containing all instances of $\varphi_1, \ldots, \varphi_n$. The intersection is non-empty as $\mathrm{Frm}(\mathcal{L})$, the set of all formulas, is such a modal logic. □

Definition prf.9. The smallest normal modal logic containing $\varphi_1, \ldots, \varphi_n$ is called a *modal system* and denoted by $\mathbf{K}\varphi_1 \ldots \varphi_n$. The smallest normal modal logic is denoted by $\mathbf{K}$.

prf.3 Derivations and Modal Systems

We first define what a derivation is for normal modal logics. Roughly, a derivation is a sequence of formulas in which every element is either (a substitution instance of) one of a number of *axioms*, or follows from previous elements by one of a few inference rules. For normal modal logics, all instances of tautologies, K, and DUAL count as axioms. This results in the modal system **K**, the smallest normal modal logic. We may wish to add additional axioms to obtain other systems, however. The rules are always modus ponens MP and necessitation NEC. mod:prf:prf:sec

Definition prf.10. Given a modal system $\mathbf{K}\varphi_1 \dots \varphi_n$ and a formula ψ we say that ψ is *derivable* in $\mathbf{K}\varphi_1 \dots \varphi_n$, written $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$, if and only if there are formulas $\chi_1, \dots, \chi_k$ such that $\chi_k = \psi$ and each χ_i is either a tautological instance, or an instance of one of K, DUAL, $\varphi_1, \dots, \varphi_n$, or it follows from previous formulas by means of the rules MP or NEC.

The following proposition allows us to show that $\psi \in \Sigma$ by exhibiting a Σ-proof of ψ.

Proposition prf.11. $\mathbf{K}\varphi_1 \dots \varphi_n = \{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\}$.

Proof. We use induction on the length of derivations to show that $\{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\} \subseteq \mathbf{K}\varphi_1 \dots \varphi_n$.

If the derivation of ψ has length 1, it contains a single formula. That formula cannot follow from previous formulas by MP or NEC, so must be a tautological instance, an instance of K, DUAL, or an instance of one of $\varphi_1, \dots, \varphi_n$. But $\mathbf{K}\varphi_1 \dots \varphi_n$ contains these as well, so $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$.

If the derivation of ψ has length > 1, then ψ may in addition be obtained by MP or NEC from formulas not occurring as the last line in the derivation. If ψ follows from χ and $\chi \to \psi$ (by MP), then χ and $\chi \to \psi \in \mathbf{K}\varphi_1 \dots \varphi_n$ by induction hypothesis. But every modal logic is closed under modus ponens, so $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$. If $\psi \equiv \Box\chi$ follows from χ by NEC, then $\chi \in \mathbf{K}\varphi_1 \dots \varphi_n$ by induction hypothesis. But every normal modal logic is closed under NEC, so $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$.

The converse inclusion follows by showing that $\Sigma = \{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\}$ is a normal modal logic containing all the instances of $\varphi_1, \dots, \varphi_n$, and the observation that $\mathbf{K}\varphi_1 \dots \varphi_n$ is, by definition, the smallest such logic.

1. Every tautology ψ is a tautological instance, so $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$, so Σ contains all tautologies.

2. If $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \chi$ and $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \chi \to \psi$, then $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$: Combine the derivation of χ with that of $\chi \to \psi$, and add the line ψ. The last line is justified by MP. So Σ is closed under modus ponens.

3. If ψ has a derivation, then every substitution instance of ψ also has a derivation: apply the substitution to every formula in the derivation.

(Exercise: prove by induction on the length of derivations that the result is also a correct derivation). So Σ is closed under uniform substitution. (We have now established that Σ satisfies all conditions of a modal logic.)

4. We have $\mathbf{K}\varphi_1 \dots \varphi_n \vdash$ K, so $K \in \Sigma$.

5. We have $\mathbf{K}\varphi_1 \dots \varphi_n \vdash$ DUAL, so DUAL $\in \Sigma$.

6. If $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \chi$, the additional line $\Box\chi$ is justified by NEC. Consequently, Σ is closed under NEC. Thus, Σ is normal.

□

prf.4 Proofs in K

mod:prf:prk: sec

In order to practice proofs in the smallest modal system, we show the valid formulas on the left-hand side of **??** can all be given **K**-proofs.

Proposition prf.12. $\mathbf{K} \vdash \Box\varphi \to \Box(\psi \to \varphi)$

Proof.

1.	$\varphi \to (\psi \to \varphi)$	TAUT
2.	$\Box(\varphi \to (\psi \to \varphi))$	NEC, 1
3.	$\Box(\varphi \to (\psi \to \varphi)) \to (\Box\varphi \to \Box(\psi \to \varphi))$	K
4.	$\Box\varphi \to \Box(\psi \to \varphi)$	MP, 2, 3

□

Proposition prf.13. $\mathbf{K} \vdash \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$

Proof.

1.	$(\varphi \wedge \psi) \to \varphi$	TAUT
2.	$\Box((\varphi \wedge \psi) \to \varphi)$	NEC
3.	$\Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\varphi)$	K
4.	$\Box(\varphi \wedge \psi) \to \Box\varphi$	MP, 2, 3
5.	$(\varphi \wedge \psi) \to \psi$	TAUT
6.	$\Box((\varphi \wedge \psi) \to \psi)$	NEC
7.	$\Box((\varphi \wedge \psi) \to \psi) \to (\Box(\varphi \wedge \psi) \to \Box\psi)$	K
8.	$\Box(\varphi \wedge \psi) \to \Box\psi$	MP, 6, 7
9.	$(\Box(\varphi \wedge \psi) \to \Box\varphi) \to$ $((\Box(\varphi \wedge \psi) \to \Box\varphi) \to$ $(\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)))$	TAUT
10.	$(\Box(\varphi \wedge \psi) \to \Box\varphi) \to$ $(\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi))$	MP, 4, 9
11.	$\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$	MP, 4, 10.

Note that the formula on line 9 is an instance of the tautology

$$(p \to q) \to ((p \to r) \to (p \to (q \wedge r))).$$

□

Proposition prf.14. $\mathbf{K} \vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$

Proof.

1.	$\varphi \to (\psi \to (\varphi \land \psi))$	TAUT
2.	$\Box(\varphi \to (\psi \to (\varphi \land \psi)))$	NEC, 1
3.	$\Box(\varphi \to (\psi \to (\varphi \land \psi))) \to (\Box\varphi \to \Box(\psi \to (\varphi \land \psi)))$	K
4.	$\Box\varphi \to \Box(\psi \to (\varphi \land \psi))$	MP, 2, 3
5.	$\Box(\psi \to (\varphi \land \psi)) \to (\Box\psi \to \Box(\varphi \land \psi))$	K
6.	$(\Box\varphi \to \Box(\psi \to (\varphi \land \psi))) \to$ $(\Box(\psi \to (\varphi \land \psi)) \to (\Box\psi \to \Box(\varphi \land \psi))) \to$ $(\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi))))$	TAUT
7.	$(\Box(\psi \to (\varphi \land \psi)) \to (\Box\psi \to \Box(\varphi \land \psi))) \to$ $(\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi)))$	MP, 4, 6
8.	$\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi)))$	MP, 5, 7
9.	$(\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi)))) \to$ $((\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi))$	TAUT
10.	$(\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$	MP, 8, 9

The formulas on lines 6 and 9 are instances of the tautologies

$$(p \to q) \to ((q \to r) \to (p \to r))$$
$$(p \to (q \to r)) \to ((p \land q) \to r)$$

□

Proposition prf.15. $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$

Proof.

1.	$\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$	DUAL
2.	$(\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p) \to$ $(\neg\Box\neg\neg p \to \Diamond\neg p)$	TAUT
3.	$\neg\Box\neg\neg p \to \Diamond\neg p$	MP, 1, 2
4.	$\neg\neg p \to p$	TAUT
5.	$\Box(\neg\neg p \to p)$	NEC, 4
6.	$\Box(\neg\neg p \to p) \to (\Box\neg\neg p \to \Box p)$	K
7.	$(\Box\neg\neg p \to \Box p)$	MP, 5, 6
8.	$(\Box\neg\neg p \to \Box p) \to (\neg\Box p \to \neg\Box\neg\neg p)$	TAUT
9.	$\neg\Box p \to \neg\Box\neg\neg p$	MP, 7, 8
10.	$(\neg\Box p \to \neg\Box\neg\neg p) \to$ $((\neg\Box\neg\neg p \to \Diamond\neg p) \to (\neg\Box p \to \Diamond\neg p))$	TAUT
11.	$(\neg\Box\neg\neg p \to \Diamond\neg p) \to (\neg\Box p \to \Diamond\neg p)$	MP, 9, 10
12.	$\neg\Box p \to \Diamond\neg p$	MP, 3, 11

The formulas on lines 8 and 10 are instances of the tautologies

$$(p \to q) \to (\neg q \to \neg p)$$
$$(p \to q) \to ((q \to r) \to (p \to r)).$$

$\square$

Problem prf.2. Find derivations in $\mathbf{K}$ for the following formulas:

1. $\Box\neg p \to \Box(p \to q)$
2. $(\Box p \lor \Box q) \to \Box(p \lor q)$
3. $\Diamond p \to \Diamond(p \lor q)$

prf.5 Derived Rules

mod:prf:der:sec

Finding and writing derivations is obviously difficult, cumbersome, and repetitive. For instance, very often we want to pass from $\varphi \to \psi$ to $\Box\varphi \to \Box\psi$, i.e., apply rule RK. That requires an application of NEC, then recording the proper instance of K, then applying MP. Passing from $\varphi \to \psi$ and $\psi \to \chi$ to $\varphi \to \chi$ requires recording the (long) tautological instance

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$

and applying MP twice. Often we want to replace a sub-formula by a formula we know to be equivalent, e.g., $\Diamond\varphi$ by $\neg\Box\neg\varphi$, or $\neg\neg\varphi$ by φ. So rather than write out the actual derivation, it is more convenient to simply record why the intermediate steps are derivable. For this purpose, let us collect some facts about derivability.

Proposition prf.16. *If $\mathbf{K} \vdash \varphi_1$, ..., $\mathbf{K} \vdash \varphi_n$, and ψ follows from φ_1, ..., φ_n by propositional logic, then $\mathbf{K} \vdash \psi$.*

Proof. If ψ follows from φ_1, ..., φ_n by propositional logic, then

$$\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \psi) \dots)$$

is a tautological instance. Applying MP n times gives a derivation of ψ. $\square$

We will indicate use of this proposition by PL.

Proposition prf.17. *If $\mathbf{K} \vdash \varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \dots)$ then $\mathbf{K} \vdash \Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n) \dots)$.*

Proof. By induction on n, just as in the proof of Proposition prf.6. $\square$

We will indicate use of this proposition by RK. Let's illustrate how these results help establishing derivability results more easily.

Proposition prf.18. $\mathbf{K} \vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$

Proof.

1.	$\mathbf{K} \vdash \varphi \to (\psi \to (\varphi \land \psi))$	TAUT
2.	$\mathbf{K} \vdash \Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi)))$	RK, 1
3.	$\mathbf{K} \vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$	PL, 2

$\Box$

Proposition prf.19. *If* $\mathbf{K} \vdash \varphi \leftrightarrow \psi$ *and* $\mathbf{K} \vdash \chi[\varphi/p]$ *then* $\mathbf{K} \vdash \chi[B/p]$ *mod:prf:der: prop:rewriting*

Proof. Exercise. $\Box$

Problem prf.3. Prove Proposition prf.19 by proving, by induction on the complexity of χ, that if $\mathbf{K} \vdash \varphi \leftrightarrow \psi$ then $\mathbf{K} \vdash \chi[\varphi/p] \leftrightarrow \chi[\psi/p]$.

This proposition comes in handy especially when we want to convert $\Diamond$ into $\Box$ (or vice versa), or remove double negations inside a formula. For instance:

Proposition prf.20. $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$

Proof.

1.	$\mathbf{K} \vdash \Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$	DUAL
2.	$\mathbf{K} \vdash \neg\Box\neg\neg p \to \Diamond\neg p$	PL, 1
3.	$\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$	re-write p for $\neg\neg p$

$\Box$

The following proposition justifies that we can establish derivability results schematically. E.g., the previous proposition does not just establish that $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$, but $\mathbf{K} \vdash \neg\Box\varphi \to \Diamond\neg\varphi$ for arbitrary φ.

Proposition prf.21. *If* φ *is a substitution instance of* ψ *and* $\mathbf{K} \vdash \psi$*, then* $\mathbf{K} \vdash \varphi$*.*

Proof. It is tedious but routine to verify (by induction on the length of the derivation of ψ) that applying a substitution to an entire derivation also results in a correct derivation. Specifically, substitution instances of tautological instances are themselves tautological instances, substitution instances of instances of DUAL and K are themselves instances of DUAL and K, and applications of MP and NEC remain correct when substituting formulas for propositional variables in both premise(s) and conclusion. $\Box$

prf.6 More Proofs in K

Let's see some more examples of derivability in $\mathbf{K}$, now using the simplified method introduced in section prf.5. *mod:prf:mpr: sec*

Proposition prf.22. $\mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$

Proof.

1. $\mathbf{K} \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ PL
2. $\mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Box\neg\psi \to \Box\neg\varphi)$ RK, 1
3. $\mathbf{K} \vdash (\Box\neg\psi \to \Box\neg\varphi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$ TAUT
4. $\mathbf{K} \vdash (\Box\neg\psi \to \Box\neg\varphi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$ PL, 2, 3
5. $\mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$ re-writing $\Diamond$ for $\neg\Box\neg$.

□

Proposition prf.23. $\mathbf{K} \vdash \Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$

Proof.

1. $\mathbf{K} \vdash \varphi \to (\neg\psi \to \neg(\varphi \to \psi))$ TAUT
2. $\mathbf{K} \vdash \Box\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \to \psi))$ RK, 1
3. $\mathbf{K} \vdash \Box\varphi \to (\neg\Box\neg(\varphi \to \psi) \to \neg\Box\neg\psi)$ PL, 2
4. $\mathbf{K} \vdash \Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$ re-writing $\Diamond$ for $\neg\Box\neg$.

□

Proposition prf.24. $\mathbf{K} \vdash (\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi)$

Proof.

1. $\mathbf{K} \vdash \neg(\varphi \lor \psi) \to \neg\varphi$ TAUT
2. $\mathbf{K} \vdash \Box\neg(\varphi \lor \psi) \to \Box\neg\varphi$ RK, 1
3. $\mathbf{K} \vdash \neg\Box\neg\varphi \to \neg\Box\neg(\varphi \lor \psi)$ PL, 2
4. $\mathbf{K} \vdash \Diamond\varphi \to \Diamond(\varphi \lor \psi)$ re-writing
5. $\mathbf{K} \vdash \Diamond\psi \to \Diamond(\varphi \lor \psi)$ similarly
6. $\mathbf{K} \vdash (\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi)$ PL, 4, 5.

□

Proposition prf.25. $\mathbf{K} \vdash \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$

Proof.

1. $\mathbf{K} \vdash \neg\varphi \to (\neg\psi \to \neg(\varphi \lor \psi)$ TAUT
2. $\mathbf{K} \vdash \Box\neg\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \lor \psi)$ RK
3. $\mathbf{K} \vdash \Box\neg\varphi \to (\neg\Box\neg(\varphi \lor \psi) \to \neg\Box\neg\psi))$ PL, 2
4. $\mathbf{K} \vdash \neg\Box\neg(\varphi \lor \psi) \to (\Box\neg\varphi \to \neg\Box\neg\psi)$ PL, 3
5. $\mathbf{K} \vdash \neg\Box\neg(\varphi \lor \psi) \to (\neg\neg\Box\neg\psi \to \neg\Box\neg\varphi)$ PL, 4
6. $\mathbf{K} \vdash \Diamond(\varphi \lor \psi) \to (\neg\Diamond\psi \to \Diamond\varphi)$ re-writing $\Diamond$ for $\neg\Box\neg$
7. $\mathbf{K} \vdash \Diamond(\varphi \lor \psi) \to (\Diamond\psi \lor \Diamond\varphi)$ PL, 6.

□

Problem prf.4. Show that the following derivability claims hold:

1. $\mathbf{K} \vdash \Diamond\neg\bot \to (\Box\varphi \to \Diamond\varphi)$;
2. $\mathbf{K} \vdash \Box(\varphi \lor \psi) \to (\Diamond\varphi \lor \Box\psi)$;
3. $\mathbf{K} \vdash (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$.

prf.7 Dual Formulas

mod:prf:dua:sec

Definition prf.26. Each of the formulas T, B, 4, and 5 has a *dual*, denoted by a subscripted diamond, as follows: mod:prf:dua:def:duals

$$p \to \Diamond p \tag{$\mathrm{T}_\Diamond$}$$

$$\Diamond\Box p \to p \tag{$\mathrm{B}_\Diamond$}$$

$$\Diamond\Diamond p \to \Diamond p \tag{$4_\Diamond$}$$

$$\Diamond\Box p \to \Box p \tag{$5_\Diamond$}$$

Each of the above dual formulas is obtained from the corresponding formula by substituting $\neg p$ for p, contraposing, replacing $\neg\Box\neg$ by $\Diamond$, and replacing $\neg\Diamond\neg$ by $\Box$. D, i.e., $\Box\varphi \to \Diamond\varphi$ is its own dual in that sense.

Problem prf.5. Show that for each formula φ in Definition prf.26: $\mathbf{K} \vdash \varphi \leftrightarrow \varphi_\Diamond$.

prf.8 Proofs in Modal Systems

We now come to proofs in systems of modal logic other than **K**. mod:prf:prs:sec

Proposition prf.27. *The following provability results obtain:* *mod:prf:prs:prop:S5facts*

1. $\mathbf{KT5} \vdash \mathrm{B}$;
2. $\mathbf{KT5} \vdash 4$;
3. $\mathbf{KDB4} \vdash \mathrm{T}$;
4. $\mathbf{KB4} \vdash 5$;
5. $\mathbf{KB5} \vdash 4$;
6. $\mathbf{KT} \vdash \mathrm{D}$. *mod:prf:prs:prop:S5facts-KT-D*

Proof. We exhibit proofs for each.

1. $\mathbf{KT5} \vdash \mathrm{B}$:

1.	$\mathbf{KT5} \vdash \Diamond\varphi \to \Box\Diamond\varphi$	5
2.	$\mathbf{KT5} \vdash \varphi \to \Diamond\varphi$	$\mathrm{T}_\Diamond$
3.	$\mathbf{KT5} \vdash \varphi \to \Box\Diamond\varphi$	PL.

2. $\mathbf{KT5} \vdash 4$:

1. $\mathbf{KT5} \vdash \Diamond\Box\varphi \to \Box\Diamond\Box\varphi$ 5 with $\Box\varphi$ for p
2. $\mathbf{KT5} \vdash \Box\varphi \to \Diamond\Box\varphi$ $\mathrm{T}_\Diamond$ with $\Box\varphi$ for p
3. $\mathbf{KT5} \vdash \Box\varphi \to \Box\Diamond\Box\varphi$ PL, 1, 2
4. $\mathbf{KT5} \vdash \Diamond\Box\varphi \to \Box\varphi$ $5_\Diamond$
5. $\mathbf{KT5} \vdash \Box\Diamond\Box\varphi \to \Box\Box\varphi$ RK, 4
6. $\mathbf{KT5} \vdash \Box\varphi \to \Box\Box\varphi$ PL, 3, 5.

3. **KDB4** $\vdash$ T:

   1. $\mathbf{KDB4} \vdash \Diamond\Box\varphi \to \varphi$ $\mathrm{B}_\Diamond$
   2. $\mathbf{KDB4} \vdash \Box\Box\varphi \to \Diamond\Box\varphi$ D with $\Box\varphi$ for p
   3. $\mathbf{KDB4} \vdash \Box\Box\varphi \to \varphi$ PL1, 2
   4. $\mathbf{KDB4} \vdash \Box\varphi \to \Box\Box\varphi$ 4
   5. $\mathbf{KDB4} \vdash \Box\varphi \to \varphi$ PL, 1, 4.

4. **KB4** $\vdash$ 5:

   1. $\mathbf{KB4} \vdash \Diamond\varphi \to \Box\Diamond\Diamond\varphi$ B with $\Diamond\varphi$ for p
   2. $\mathbf{KB4} \vdash \Diamond\Diamond\varphi \to \Diamond\varphi$ $4_\Diamond$
   3. $\mathbf{KB4} \vdash \Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi$ RK, 2
   4. $\mathbf{KB4} \vdash \Diamond\varphi \to \Box\Diamond\varphi$ PL, 1, 3.

5. **KB5** $\vdash$ 4:

   1. $\mathbf{KB5} \vdash \Box\varphi \to \Box\Diamond\Box\varphi$ B with $\Box\varphi$ for p
   2. $\mathbf{KB5} \vdash \Diamond\Box\varphi \to \Box\varphi$ $5_\Diamond$
   3. $\mathbf{KB5} \vdash \Box\Diamond\Box\varphi \to \Box\Box\varphi$ RK, 2
   4. $\mathbf{KB5} \vdash \Box\varphi \to \Box\Box\varphi$ PL, 1, 3.

6. **KT** $\vdash$ D:

   1. $\mathbf{KT} \vdash \Box\varphi \to \varphi$ T
   2. $\mathbf{KT} \vdash \varphi \to \Diamond\varphi$ $\mathrm{T}_\Diamond$
   3. $\mathbf{KT} \vdash \Box\varphi \to \Diamond\varphi$ PL, 1, 2

□

Definition prf.28. Following tradition, we define **S4** to be the system **KT4**, and **S5** the system **KTB4**.

The following proposition shows that the classical system **S5** has several equivalent axiomatizations. This should not surprise, as the various combinations of axioms all characterize equivalence relations (see **??**).

mod:prf:prs: prop:S5 **Proposition prf.29.** **KTB4** = **KT5** = **KDB4** = **KDB5**.

Proof. Exercise. □

Problem prf.6. Prove Proposition prf.29.

prf.9 Soundness

A derivation system is called sound if everything that can be derived is valid. When considering modal systems, i.e., derivations where in addition to K we can use instances of some formulas $\varphi_1, \dots, \varphi_n$, we want every derivable formula to be true in any model in which $\varphi_1, \dots, \varphi_n$ are true. mod:prf:snd:sec

Theorem prf.30 (Soundness Theorem)**.** *If every instance of $\varphi_1, \dots, \varphi_n$ is valid in the classes of models $\mathcal{C}_1, \dots, \mathcal{C}_n$, respectively, then $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$ implies that ψ is valid in the class of models $\mathcal{C}_1 \cap \dots \cap \mathcal{C}_n$.* mod:prf:snd:thm:soundness

Proof. By induction on length of proofs. For brevity, put $\mathcal{C} = \mathcal{C}_n \cap \dots \cap \mathcal{C}_n$.

1. Induction Basis: If ψ has a proof of length 1, then it is either a tautological instance, an instance of K, or of DUAL, or an instance of one of $\varphi_1, \dots, \varphi_n$. In the first case, ψ is valid in $\mathcal{C}$, since tautological instance are valid in *any* class of models, by **??**. Similarly in the second case, by **??** and **??**. Finally in the third case, since ψ is valid in $\mathcal{C}_i$ and $\mathcal{C} \subseteq \mathcal{C}_i$, we have that ψ is valid in $\mathcal{C}$ as well.

2. Inductive step: Suppose ψ has a proof of length $k > 1$. If ψ is a tautological instance or an instance of one of $\varphi_1, \dots, \varphi_n$, we proceed as in the previous step. So suppose ψ is obtained by MP from previous formulas $\chi \to \psi$ and χ. Then $\chi \to \psi$ and χ have proofs of length $< k$, and by inductive hypothesis they are valid in $\mathcal{C}$. By **??**, ψ is valid in $\mathcal{C}$ as well. Finally suppose ψ is obtained by NEC from χ (so that $\psi = \Box\chi$). By inductive hypothesis, χ is valid in $\mathcal{C}$, and by **??** so is ψ. □

prf.10 Showing Systems are Distinct

In section prf.8 we saw how to prove that two systems of modal logic are in fact the same system. Theorem prf.30 allows us to show that two modal systems Σ and Σ' are distinct, by finding a formula φ such that $\Sigma' \vdash \varphi$ that fails in a model of Σ. mod:prf:dis:sec

Proposition prf.31. $\mathbf{KD} \subsetneq \mathbf{KT}$

Proof. This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show $\mathbf{KD} \subseteq \mathbf{KT}$ we need to see that $\mathbf{KD} \vdash \psi$ implies $\mathbf{KT} \vdash \psi$, which follows from $\mathbf{KT} \vdash \mathbf{D}$, as shown in Proposition prf.27(6). To show that the inclusion is proper, by Soundness (Theorem prf.30), it suffices to exhibit a model of $\mathbf{KD}$ where T, i.e., $\Box p \to p$, fails (an easy task left as an exercise), for then by Soundness $\mathbf{KD} \nvdash \Box p \to p$. □

Proposition prf.32. $\mathbf{KB} \neq \mathbf{K4}$.

Proof. We construct a symmetric model where some instance of 4 fails; since obviously the instance is derivable for **K4** but not in **KB**, it will follow **K4** $\not\subseteq$ **KB**. Consider the symmetric model $\mathfrak{M}$ of Figure prf.1. Since the model is symmetric, K and B are true in $\mathfrak{M}$ (by **??** and **??**, respectively). However, $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$. □

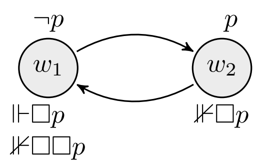

Figure prf.1: A symmetric model falsifying an instance of 4.

mod:prf:dis: figKBnot4 thm:KTBnot45

Theorem prf.33. **KTB** $\nvdash$ **4** *and* **KTB** $\nvdash$ **5**.

Proof. By **??** we know that all instances of T and B are true in every reflexive symmetric model (respectively). So by soundness, it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figure prf.2. Then $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$, so 4 fails at w_1. Similarly, $\mathfrak{M}, w_2 \nVdash \Diamond\neg p \to \Box\Diamond\neg p$, so the instance of 5 with $\varphi = \neg p$ fails at w_2. □

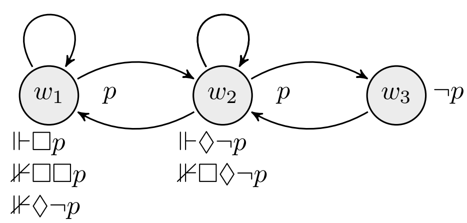

Figure prf.2: The model for Theorem prf.33.

mod:prf:dis: figKTBnot45 thm:KD5not4

Theorem prf.34. **KD5** $\neq$ **KT4** $=$ **S4**.

Proof. By **??** we know that all instances of D and 5 are true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figure prf.3, and notice that $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$. □

Problem prf.7. Give an alternative proof of Theorem prf.34 using a model with 3 worlds.

Problem prf.8. Provide a single reflexive transitive model showing that both **KT4** $\nvdash$ B and **KT4** $\nvdash$ 5.

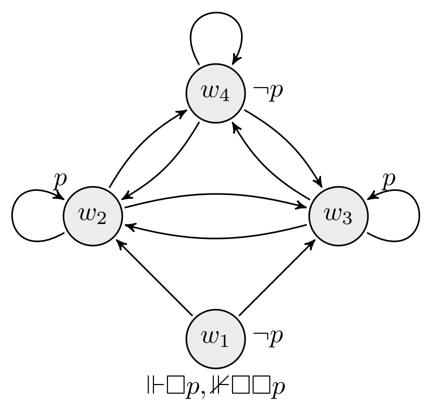

Figure prf.3: The model for Theorem prf.34.

mod:prf:dis:fig:KD5not4

prf.11 Derivability from a Set of Formulas

In section prf.8 we defined a notion of provability of a formula in a system Σ. We now extend this notion to provability in Σ from formulas in a set Γ. mod:prf:prg:sec

Definition prf.35. A formula φ is derivable in a system Σ from a set of formulas Γ, written $\Gamma \vdash_\Sigma \varphi$ if and only if there are $\psi_1, \dots, \psi_n \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$. mod:prf:prg:defn:Gammaproves

prf.12 Properties of Derivability

mod:prf:prp:sec

Proposition prf.36. *Let Σ be a modal system and Γ a set of modal formulas. The following properties hold:* *mod:prf:prp:prop:derivabilityfacts*

1. Monotony*: If $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_\Sigma \varphi$;*
2. Reflexivity*: If $\varphi \in \Gamma$ then $\Gamma \vdash_\Sigma \varphi$;*
3. Cut*: If $\Gamma \vdash_\Sigma \varphi$ and $\Delta \cup \{\varphi\} \vdash_\Sigma \psi$ then $\Gamma \cup \Delta \vdash_\Sigma \psi$;*
4. Deduction theorem*: $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi$ if and only if $\Gamma \vdash_\Sigma \psi \to \varphi$;*
5. Rule T*: If $\Gamma \vdash_\Sigma \varphi_1$ and … and $\Gamma \vdash_\Sigma \varphi_n$ and $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \psi) \cdots)$ is a tautological instance, then $\Gamma \vdash_\Sigma \psi$.* *mod:prf:prp:prop:derivabilityfacts-ruleT*

The proof is an easy exercise. Part (5) of Proposition prf.36 gives us that, for instance, if $\Gamma \vdash_\Sigma \varphi \lor \psi$ and $\Gamma \vdash_\Sigma \neg\varphi$, then $\Gamma \vdash_\Sigma \psi$. Also, in what follows, we write $\Gamma, \varphi \vdash_\Sigma \psi$ instead of $\Gamma \cup \{\varphi\} \vdash_\Sigma \psi$.

Definition prf.37. A set Γ is *deductively closed* relatively to a system Σ if and only if $\Gamma \vdash_\Sigma \varphi$ implies $\varphi \in \Gamma$.

prf.13 Consistency

mod:prf:con: sec

Consistency is an important property of sets of formulas. A set of formulas is inconsistent if a contradiction, such as $\bot$, is derivable from it; and otherwise consistent. If a set is inconsistent, its formulas cannot all be true in a model at a world. For the completeness theorem we prove the converse: every consistent set is true at a world in a model, namely in the "canonical model."

Definition prf.38. A set Γ is *consistent* relatively to a system Σ or, as we will say, Σ-consistent, if and only if $\Gamma \nvdash_{\Sigma} \bot$.

So for instance, the set $\{\Box(p \to q), \Box p, \neg\Box q\}$ is consistent relatively to propositional logic, but not **K**-consistent. Similarly, the set $\{\Diamond p, \Box\Diamond p \to q, \neg q\}$ is not **K5**-consistent.

mod:prf:con: prop:consistencyfacts

Proposition prf.39. *Let Γ be a set of formulas. Then:*

1. *A set Γ is Σ-consistent if and only if there is some formula φ such that $\Gamma \nvdash_{\Sigma} \varphi$.*

mod:prf:con: prop:consistencyfacts-b

2. *$\Gamma \vdash_{\Sigma} \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is not Σ-consistent.*

mod:prf:con: prop:consistencyfacts-c

3. *If Γ is Σ-consistent, then for any formula φ, either $\Gamma \cup \{\varphi\}$ is Σ-consistent or $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent.*

Proof. These facts follow easily using classical propositional logic. We give the argument for (3). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent. Then by (2), both $\Gamma, \varphi \vdash_{\Sigma} \bot$ and $\Gamma, \neg\varphi \vdash_{\Sigma} \bot$. By the deduction theorem $\Gamma \vdash_{\Sigma} \varphi \to \bot$ and $\Gamma \vdash_{\Sigma} \neg\varphi \to \bot$. But $(\varphi \to \bot) \to ((\neg\varphi \to \bot) \to \bot)$ is a tautological instance, hence by Proposition prf.36(5), $\Gamma \vdash_{\Sigma} \bot$. $\Box$

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Bibliography