mar.1 Standard Models of Arithmetic

mod:mar:stm: sec The language of arithmetic $\mathcal{L}_A$ is obviously intended to be about numbers, specifically, about natural numbers. So, "the" standard model $\mathfrak{N}$ is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two structures that are isomorphic share those. So we can be a bit more liberal, and consider any structure that is isomorphic to $\mathfrak{N}$ "standard."

Definition mar.1. A structure for $\mathcal{L}_A$ is *standard* if it is isomorphic to $\mathfrak{N}$.

mod:mar:stm: prop:standard-domain **Proposition mar.2.** *If a structure $\mathfrak{M}$ is standard, then its domain is the set of values of the standard numerals, i.e.,*

$$|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$$

Proof. Clearly, every $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \in |\mathfrak{M}|$. We just have to show that every $x \in |\mathfrak{M}|$ is equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some n. Since $\mathfrak{M}$ is standard, it is isomorphic to $\mathfrak{N}$. Suppose $g \colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism. Then $g(n) = g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. But for every $x \in |\mathfrak{M}|$, there is an $n \in \mathbb{N}$ such that $g(n) = x$, since g is surjective. □

If a structure $\mathfrak{M}$ for $\mathcal{L}_A$ is standard, the elements of its domain can all be named by the standard numerals $\overline{0}$, $\overline{1}$, $\overline{2}$, …, i.e., the terms o, o', o'', etc. Of course, this does not mean that the elements of $|\mathfrak{M}|$ *are* the numbers, just that we can pick them out the same way we can pick out the numbers in $|\mathfrak{N}|$. explanation

Problem mar.1. Show that the converse of Proposition mar.2 is false, i.e., give an example of a structure $\mathfrak{M}$ with $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$ that is not isomorphic to $\mathfrak{N}$.

mod:mar:stm: prop:thq-standard **Proposition mar.3.** *If $\mathfrak{M} \vDash \mathbf{Q}$, and $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$, then $\mathfrak{M}$ is standard.*

Proof. We have to show that $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$. Consider the function $g \colon \mathbb{N} \to |\mathfrak{M}|$ defined by $g(n) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. By the hypothesis, g is surjective. It is also injective: $\mathbf{Q} \vdash \overline{n} \neq \overline{m}$ whenever $n \neq m$. Thus, since $\mathfrak{M} \vDash \mathbf{Q}$, $\mathfrak{M} \vDash \overline{n} \neq \overline{m}$, whenever $n \neq m$. Thus, if $n \neq m$, then $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \neq \mathrm{Val}^{\mathfrak{M}}(\overline{m})$, i.e., $g(n) \neq g(m)$.

We also have to verify that g is an isomorphism.

1. We have $g(\mathsf{o}^{\mathfrak{N}}) = g(0)$ since, $\mathsf{o}^{\mathfrak{N}} = 0$. By definition of g, $g(0) = \mathrm{Val}^{\mathfrak{M}}(\overline{0})$. But $\overline{0}$ is just o, and the value of a term which happens to be a constant symbol is given by what the structure assigns to that constant symbol, i.e., $\mathrm{Val}^{\mathfrak{M}}(\mathsf{o}) = \mathsf{o}^{\mathfrak{M}}$. So we have $g(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$ as required.
2. $g({\prime}^{\mathfrak{N}}(n)) = g(n+1)$, since $\prime$ in $\mathfrak{N}$ is the successor function on $\mathbb{N}$. Then, $g(n+1) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+1})$ by definition of g. But $\overline{n+1}$ is the same term as $\overline{n}'$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n}')$. By the definition of the value function, this is $= {\prime}^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ we get $g({\prime}^{\mathfrak{N}}(n)) = {\prime}^{\mathfrak{M}}(g(n))$.

3. $g(+^{\mathfrak{N}}(n, m)) = g(n + m)$, since $+$ in $\mathfrak{N}$ is the addition function on $\mathbb{N}$. Then, $g(n + m) = \mathrm{Val}^{\mathfrak{M}}(\overline{n + m})$ by definition of g. But $\mathbf{Q} \vdash \overline{n + m} = (\overline{n} + \overline{m})$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n + m}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n} + \overline{m})$. By the definition of the value function, this is $= +^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ and $\mathrm{Val}^{\mathfrak{M}}(\overline{m}) = g(m)$, we get $g(+^{\mathfrak{N}}(n, m)) = +^{\mathfrak{M}}(g(n), g(m))$.

4. $g(\times^{\mathfrak{N}}(n, m)) = \times^{\mathfrak{M}}(g(n), g(m))$: Exercise.

5. $\langle n, m\rangle \in <^{\mathfrak{N}}$ iff $n < m$. If $n < m$, then $\mathbf{Q} \vdash \overline{n} < \overline{m}$, and also $\mathfrak{M} \vDash \overline{n} < \overline{m}$. Thus $\langle \mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m})\rangle \in <^{\mathfrak{M}}$, i.e., $\langle g(n), g(m)\rangle \in <^{\mathfrak{M}}$. If $n \not< m$, then $\mathbf{Q} \vdash \neg\overline{n} < \overline{m}$, and consequently $\mathfrak{M} \nvDash \overline{n} < \overline{m}$. Thus, as before, $\langle g(n), g(m)\rangle \notin <^{\mathfrak{M}}$. Together, we get: $\langle n, m\rangle \in <^{\mathfrak{N}}$ iff $\langle g(n), g(m)\rangle \in <^{\mathfrak{M}}$. □

explanation

The function g is the most obvious way of defining a mapping from $\mathbb{N}$ to the domain of any other structure $\mathfrak{M}$ for $\mathcal{L}_A$, since every such $\mathfrak{M}$ contains elements named by $\overline{0}$, $\overline{1}$, $\overline{2}$, etc. So it isn't surprising that if $\mathfrak{M}$ makes at least some basic statements about the $\overline{n}$'s true in the same way that $\mathfrak{N}$ does, and g is also bijective, then g will turn into an isomorphism. In fact, if $|\mathfrak{M}|$ contains no elements other than what the $\overline{n}$'s name, it's the only one.

Proposition mar.4. *If $\mathfrak{M}$ is standard, then g from the proof of Proposition mar.3 is the only isomorphism from $\mathfrak{N}$ to $\mathfrak{M}$.* mod:mar:stm:prop:thq-unique-iso

Proof. Suppose $h \colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism between $\mathfrak{N}$ and $\mathfrak{M}$. We show that $g = h$ by induction on n. If $n = 0$, then $g(0) = \mathsf{o}^{\mathfrak{M}}$ by definition of g. But since h is an isomorphism, $h(0) = h(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$, so $g(0) = h(0)$.

Now consider the case for $n + 1$. We have

$$\begin{aligned}
g(n+1) &= \mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) \text{ by definition of } g\\
&= \mathrm{Val}^{\mathfrak{M}}(\overline{n}') \text{ since } \overline{n+1} \equiv \overline{n}'\\
&= {\prime}^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n})) \text{ by definition of } \mathrm{Val}^{\mathfrak{M}}(t')\\
&= {\prime}^{\mathfrak{M}}(g(n)) \text{ by definition of } g\\
&= {\prime}^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis}\\
&= h({\prime}^{\mathfrak{N}}(n)) \text{ since } h \text{ is an isomorphism}\\
&= h(n+1)
\end{aligned}$$

□

explanation

For any denumerable set M, there's a bijection between $\mathbb{N}$ and M, so every such set M is potentially the domain of a standard model $\mathfrak{M}$. In fact, once you pick an object $z \in M$ and a suitable function s as $\mathsf{o}^{\mathfrak{M}}$ and ${\prime}^{\mathfrak{M}}$, the interpretations of $+$, $\times$, and $<$ is already fixed. Only functions $s \colon M \to M \setminus \{z\}$ that are both injective and surjective are suitable in a standard model as ${\prime}^{\mathfrak{M}}$. The range of s cannot contain z, since otherwise $\forall x\, \mathsf{o} \neq x'$ would be false. That sentence is true in $\mathfrak{N}$, and so $\mathfrak{M}$ also has to make it true. The function s has to be injective, since the successor function ${\prime}^{\mathfrak{N}}$ in $\mathfrak{N}$ is, and that ${\prime}^{\mathfrak{N}}$ is injective is expressed by a sentence true in $\mathfrak{N}$. It has to be surjective because otherwise there would be some $x \in M \setminus \{z\}$ not in the domain of s, i.e., the sentence $\forall x\,(x = \mathsf{o} \lor \exists y\, y' = x)$ would be false in $\mathfrak{M}$—but it is true in $\mathfrak{N}$.

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Bibliography