

## mar.1 Standard Models of Arithmetic

mod:mar:stm:  
sec

The language of arithmetic  $\mathcal{L}_A$  is obviously intended to be about numbers, specifically, about natural numbers. So, “the” standard model  $\mathfrak{N}$  is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two structures that are isomorphic share those. So we can be a bit more liberal, and consider any structure that is isomorphic to  $\mathfrak{N}$  “standard.”

**Definition mar.1.** A structure for  $\mathcal{L}_A$  is *standard* if it is isomorphic to  $\mathfrak{N}$ .

mod:mar:stm:  
prop:standard-domain

**Proposition mar.2.** If a structure  $\mathfrak{M}$  is standard, its domain is the set of values of the standard numerals, i.e.,

$$|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$$

*Proof.* Clearly, every  $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$ . We just have to show that every  $x \in |\mathfrak{M}|$  is equal to  $\text{Val}^{\mathfrak{M}}(\bar{n})$  for some  $n$ . Since  $\mathfrak{M}$  is standard, it is isomorphic to  $\mathfrak{N}$ . Suppose  $g: \mathbb{N} \rightarrow |\mathfrak{M}|$  is an isomorphism. Then  $g(n) = g(\text{Val}^{\mathfrak{N}}(\bar{n})) = \text{Val}^{\mathfrak{M}}(\bar{n})$ . But for every  $x \in |\mathfrak{M}|$ , there is an  $n \in \mathbb{N}$  such that  $g(n) = x$ , since  $g$  is surjective.  $\square$

If a structure  $\mathfrak{M}$  for  $\mathcal{L}_A$  is standard, the elements of its domain can all be named by the standard numerals  $\bar{0}, \bar{1}, \bar{2}, \dots$ , i.e., the terms  $o, o', o''$ , etc. Of course, this does not mean that the elements of  $|\mathfrak{M}|$  are the numbers, just that we can pick them out the same way we can pick out the numbers in  $|\mathfrak{N}|$ . explanation

**Problem mar.1.** Show that the converse of Proposition mar.2 is false, i.e., give an example of a structure  $\mathfrak{M}$  with  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$  that is not isomorphic to  $\mathfrak{N}$ .

mod:mar:stm:  
prop:thq-standard

**Proposition mar.3.** If  $\mathfrak{M} \models \mathbf{Q}$ , and  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ , then  $\mathfrak{M}$  is standard.

*Proof.* We have to show that  $\mathfrak{M}$  is isomorphic to  $\mathfrak{N}$ . Consider the function  $g: \mathbb{N} \rightarrow |\mathfrak{M}|$  defined by  $g(n) = \text{Val}^{\mathfrak{M}}(\bar{n})$ . By the hypothesis,  $g$  is surjective. It is also injective:  $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$  whenever  $n \neq m$ . Thus, since  $\mathfrak{M} \models \mathbf{Q}$ ,  $\mathfrak{M} \models \bar{n} \neq \bar{m}$ , whenever  $n \neq m$ . Thus, if  $n \neq m$ , then  $\text{Val}^{\mathfrak{M}}(\bar{n}) \neq \text{Val}^{\mathfrak{M}}(\bar{m})$ , i.e.,  $g(n) \neq g(m)$ .

We also have to verify that  $g$  is an isomorphism.

1. We have  $g(o^{\mathfrak{N}}) = g(0)$  since,  $o^{\mathfrak{N}} = 0$ . By definition of  $g$ ,  $g(0) = \text{Val}^{\mathfrak{M}}(\bar{0})$ . But  $\bar{0}$  is just  $o$ , and the value of a term which happens to be a constant symbol is given by what the structure assigns to that constant symbol, i.e.,  $\text{Val}^{\mathfrak{M}}(o) = o^{\mathfrak{M}}$ . So we have  $g(o^{\mathfrak{N}}) = o^{\mathfrak{M}}$  as required.
2.  $g(o^{\mathfrak{N}}(n)) = g(n+1)$ , since  $\prime$  in  $\mathfrak{N}$  is the successor function on  $\mathbb{N}$ . Then,  $g(n+1) = \text{Val}^{\mathfrak{M}}(\overline{n+1})$  by definition of  $g$ . But  $\overline{n+1}$  is the same term

as  $\bar{n}'$ , so  $\text{Val}^{\mathfrak{M}}(\overline{n+1}) = \text{Val}^{\mathfrak{M}}(\bar{n}')$ . By the definition of the value function, this is  $= \iota^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n}))$ . Since  $\text{Val}^{\mathfrak{M}}(\bar{n}) = g(n)$  we get  $g(\iota^{\mathfrak{N}}(n)) = \iota^{\mathfrak{M}}(g(n))$ .

3.  $g(+^{\mathfrak{N}}(n, m)) = g(n + m)$ , since  $+$  in  $\mathfrak{N}$  is the addition function on  $\mathbb{N}$ . Then,  $g(n + m) = \text{Val}^{\mathfrak{M}}(\overline{n + m})$  by definition of  $g$ . But  $\mathbf{Q} \vdash \overline{n + m} = \overline{(n + m)}$ , so  $\text{Val}^{\mathfrak{M}}(\overline{n + m}) = \text{Val}^{\mathfrak{M}}(\overline{n + m})$ . By the definition of the value function, this is  $= +^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n}), \text{Val}^{\mathfrak{M}}(\bar{m}))$ . Since  $\text{Val}^{\mathfrak{M}}(\bar{n}) = g(n)$  and  $\text{Val}^{\mathfrak{M}}(\bar{m}) = g(m)$ , we get  $g(+^{\mathfrak{N}}(n, m)) = +^{\mathfrak{M}}(g(n), g(m))$ .
4.  $g(\times^{\mathfrak{N}}(n, m)) = \times^{\mathfrak{M}}(g(n), g(m))$ : Exercise.
5.  $\langle n, m \rangle \in <^{\mathfrak{N}}$  iff  $n < m$ . If  $n < m$ , then  $\mathbf{Q} \vdash \bar{n} < \bar{m}$ , and also  $\mathfrak{M} \models \bar{n} < \bar{m}$ . Thus  $\langle \text{Val}^{\mathfrak{M}}(\bar{n}), \text{Val}^{\mathfrak{M}}(\bar{m}) \rangle \in <^{\mathfrak{M}}$ , i.e.,  $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$ . If  $n \not< m$ , then  $\mathbf{Q} \vdash \neg \bar{n} < \bar{m}$ , and consequently  $\mathfrak{M} \not\models \bar{n} < \bar{m}$ . Thus, as before,  $\langle g(n), g(m) \rangle \notin <^{\mathfrak{M}}$ . Together, we get:  $\langle n, m \rangle \in <^{\mathfrak{N}}$  iff  $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$ .

□

explanation

The function  $g$  is the most obvious way of defining a mapping from  $\mathbb{N}$  to the domain of any other structure  $\mathfrak{M}$  for  $\mathcal{L}_A$ , since every such  $\mathfrak{M}$  contains elements named by  $\bar{0}, \bar{1}, \bar{2}$ , etc. So it isn't surprising that if  $\mathfrak{M}$  makes at least some basic statements about the  $\bar{n}$ 's true in the same way that  $\mathfrak{N}$  does, and  $g$  is also bijective, then  $g$  will turn into an isomorphism. In fact, if  $|\mathfrak{M}|$  contains no elements other than what the  $\bar{n}$ 's name, it's the only one.

**Proposition mar.4.** *If  $\mathfrak{M}$  is standard, then  $g$  from the proof of Proposition mar.3 is the only isomorphism from  $\mathfrak{N}$  to  $\mathfrak{M}$ .*

mod:mar:stm:  
prop:thq-unique-iso

*Proof.* Suppose  $h: \mathbb{N} \rightarrow |\mathfrak{M}|$  is an isomorphism between  $\mathfrak{N}$  and  $\mathfrak{M}$ . We show that  $g = h$  by induction on  $n$ . If  $n = 0$ , then  $g(0) = o^{\mathfrak{M}}$  by definition of  $g$ . But since  $h$  is an isomorphism,  $h(0) = h(o^{\mathfrak{N}}) = o^{\mathfrak{M}}$ , so  $g(0) = h(0)$ .

Now consider the case for  $n + 1$ . We have

$$\begin{aligned}
 g(n + 1) &= \text{Val}^{\mathfrak{M}}(\overline{n + 1}) \text{ by definition of } g \\
 &= \text{Val}^{\mathfrak{M}}(\bar{n}') \\
 &= \iota^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n})) \\
 &= \iota^{\mathfrak{M}}(g(n)) \text{ by definition of } g \\
 &= \iota^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis} \\
 &= h(\iota^{\mathfrak{N}}(n)) \text{ since } h \text{ is an isomorphism} \\
 &= h(n + 1)
 \end{aligned}$$

□

For any **denumerable** set  $X$ , there's a **bijection** between  $\mathbb{N}$  and  $X$ , so every explanation such set  $X$  is potentially the **domain** of a standard model. In fact, once you pick an object  $z \in X$  and a suitable function  $s: X \rightarrow X$  as  $0^{\mathfrak{X}}$  and  $'^{\mathfrak{X}}$ , the interpretation of  $+$ ,  $\times$ , and  $<$  is already fixed. Only functions  $s = /^{\mathfrak{X}}$  that are both **injective** and **surjective** are suitable in a standard model. It has to be **injective** since the successor function in  $\mathfrak{N}$  is, and that  $'$  is **injective** is expressed by a **sentence** true in  $\mathfrak{N}$  which  $\mathfrak{X}$  thus also has to make true. It has to be **surjective** because otherwise there would be some  $x \in X$  not in the domain of  $s$ , i.e., the **sentence**  $\forall x \exists y y' = x$  would be false—but it is true in  $\mathfrak{N}$ .

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## Bibliography