mar.1  Non-Standard Models

We call a structure for $L_A$ standard if it is isomorphic to $\mathcal{N}$. If a structure isn’t isomorphic to $\mathcal{N}$, it is called non-standard.

**Definition mar.1.** A structure $\mathcal{M}$ for $L_A$ is non-standard if it is not isomorphic to $\mathcal{N}$. The elements $x \in |\mathcal{M}|$ which are equal to $\text{Val}_M(n)$ for some $n \in \mathbb{N}$ are called standard numbers (of $\mathcal{M}$), and those not, non-standard numbers.

By ??, any standard structure for $L_A$ contains only standard elements. Consequently, a non-standard structure must contain at least one non-standard element. In fact, the existence of a non-standard element guarantees that the structure is non-standard.

**Proposition mar.2.** If a structure $\mathcal{M}$ for $L_A$ contains a non-standard number, $\mathcal{M}$ is non-standard.

**Proof.** Suppose not, i.e., suppose $\mathcal{M}$ standard but contains a non-standard number $x$. Let $g: \mathbb{N} \to |\mathcal{M}|$ be an isomorphism. It is easy to see (by induction on $n$) that $g(\text{Val}_N(n)) = \text{Val}_M(n)$. In other words, $g$ maps standard numbers of $\mathcal{N}$ to standard numbers of $\mathcal{M}$. If $\mathcal{M}$ contains a non-standard number, $g$ cannot be surjective, contrary to hypothesis.

**Problem mar.1.** Recall that $Q$ contains the axioms

\begin{align*}
\forall x \forall y (x' = y' \rightarrow x = y) & \quad (Q_1) \\
\forall x (x \neq x') & \quad (Q_2) \\
\forall x (x = 0 \lor \exists y x = y') & \quad (Q_3)
\end{align*}

Give structures $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ such that

1. $\mathcal{M}_1 \models Q_1$, $\mathcal{M}_1 \models Q_2$, $\mathcal{M}_1 \not\models Q_3$;
2. $\mathcal{M}_2 \models Q_1$, $\mathcal{M}_2 \not\models Q_2$, $\mathcal{M}_2 \models Q_3$; and
3. $\mathcal{M}_3 \not\models Q_1$, $\mathcal{M}_3 \models Q_2$, $\mathcal{M}_3 \models Q_3$;

Obviously, you just have to specify $\varphi_{\mathcal{M}_i}$ for each.

It is easy enough to specify non-standard structures for $L_A$. For instance, take the structure with domain $\mathbb{Z}$ and interpret all non-logical symbols as usual. Since negative numbers are not values of $\pi$ for any $n$, this structure is non-standard. Of course, it will not be a model of arithmetic in the sense that it makes the same sentences true as $\mathcal{N}$. For instance, $\forall x x' \neq 0$ is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

**Proposition mar.3.** Let $TA = \{ \varphi : \mathcal{N} \models \varphi \}$ be the theory of $\mathcal{N}$. $TA$ has an enumerable non-standard model.
Proof. Expand \( \mathcal{L}_A \) by a new constant symbol \( c \) and consider the set of sentences
\[
\Gamma = \mathcal{T}A \cup \{ c \neq 0, c \neq 1, c \neq 2, \ldots \}
\]
Any model \( \mathcal{M}^c \) of \( \Gamma \) would contain an element \( x = x^\mathcal{M} \), which is non-standard, since \( x \neq \text{Val}^\mathcal{M}(\pi) \) for all \( n \in \mathbb{N} \). Also, obviously, \( \mathcal{M}^c \models \mathcal{T}A \), since \( \mathcal{T}A \subseteq \Gamma \). If we turn \( \mathcal{M}^c \) into a structure \( \mathcal{M} \) for \( \mathcal{L}_A \) simply by forgetting about \( c \), its domain still contains the non-standard \( x \), and also \( \mathcal{M} \models \mathcal{T}A \). The latter is guaranteed since \( c \) does not occur in \( \mathcal{T}A \). So, it suffices to show that \( \Gamma \) has a model.

We use the compactness theorem to show that \( \Gamma \) has a model. If every finite subset of \( \Gamma \) is satisfiable, so is \( \Gamma \). Consider any finite subset \( \Gamma_0 \subseteq \Gamma \). \( \Gamma_0 \) includes some sentences of \( \mathcal{T}A \) and some of the form \( c \neq \pi \), but only finitely many. Suppose \( k \) is the largest number so that \( c \neq k \in \Gamma_0 \). Define \( \mathcal{M}_k \) by expanding \( \mathcal{M} \) to include the interpretation \( c^\mathcal{M}_k = k + 1 \). \( \mathcal{M}_k \models \Gamma_0 \): if \( \varphi \in \mathcal{T}A \), \( \mathcal{M}_k \models \varphi \) since \( \mathcal{M}_k \) is just like \( \mathcal{M} \) in all respects except \( c \), and \( c \) does not occur in \( \varphi \). And \( \mathcal{M}_k \models c \neq \pi \), since \( n \leq k \), and \( \text{Val}^{\mathcal{M}_k}(c) = k + 1 \). Thus, every finite subset of \( \Gamma \) is satisfiable. \( \square \)

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Bibliography