mar.1 Non-Standard Models

We call a structure for \( \mathcal{L}_A \) standard if it is isomorphic to \( \mathfrak{N} \). If a structure isn’t isomorphic to \( \mathfrak{N} \), it is called non-standard.

**Definition mar.1.** A structure \( \mathfrak{M} \) for \( \mathcal{L}_A \) is non-standard if it is not isomorphic to \( \mathfrak{N} \). The elements \( x \in |\mathfrak{M}| \) which are equal to \( \text{Val}_{\mathfrak{M}}(n) \) for some \( n \in \mathbb{N} \) are called standard numbers (of \( \mathfrak{M} \)), and those not, non-standard numbers.

By ??, any standard structure for \( \mathcal{L}_A \) contains only standard elements. Consequently, a non-standard structure must contain at least one non-standard element. In fact, the existence of a non-standard element guarantees that the structure is non-standard.

**Proposition mar.2.** If a structure \( \mathfrak{M} \) for \( \mathcal{L}_A \) contains a non-standard number, \( \mathfrak{M} \) is non-standard.

**Proof.** Suppose not, i.e., suppose \( \mathfrak{M} \) standard but contains a non-standard number \( x \). Let \( g: \mathbb{N} \to |\mathfrak{M}| \) be an isomorphism. It is easy to see (by induction on \( n \)) that \( g(\text{Val}^\mathfrak{N}(n)) = \text{Val}^\mathfrak{M}(n) \). In other words, \( g \) maps standard numbers of \( \mathfrak{N} \) to standard numbers of \( \mathfrak{M} \). If \( \mathfrak{M} \) contains a non-standard number, \( g \) cannot be surjective, contrary to hypothesis.

**Problem mar.1.** Recall that \( \mathbb{Q} \) contains the axioms
\[
\forall x \forall y (x' = y' \to x = y) \quad (Q_1)
\]
\[
\forall x x \neq x' \quad (Q_2)
\]
\[
\forall x (x = 0 \lor \exists y x = y') \quad (Q_3)
\]

Give structures \( \mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3 \) such that

1. \( \mathfrak{M}_1 \models Q_1, \mathfrak{M}_1 \models Q_2, \mathfrak{M}_1 \not\models Q_3 \);
2. \( \mathfrak{M}_2 \models Q_1, \mathfrak{M}_2 \not\models Q_2, \mathfrak{M}_2 \models Q_3 \); and
3. \( \mathfrak{M}_3 \not\models Q_1, \mathfrak{M}_3 \models Q_2, \mathfrak{M}_3 \models Q_3 \);

Obviously, you just have to specify \( \mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3 \) for each.

It is easy enough to specify non-standard structures for \( \mathcal{L}_A \). For instance, take the structure with domain \( \mathbb{Z} \) and interpret all non-logical symbols as usual. Since negative numbers are not values of \( \bar{\pi} \) for any \( n \), this structure is non-standard. Of course, it will not be a model of arithmetic in the sense that it makes the same sentences true as \( \mathfrak{N} \). For instance, \( \forall x x' \neq 0 \) is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

**Proposition mar.3.** Let \( \text{TA} = \{ \varphi : \mathfrak{N} \models \varphi \} \) be the theory of \( \mathfrak{N} \). \( \text{TA} \) has an enumerable non-standard model.
Proof. Expand $\mathcal{L}_A$ by a new constant symbol $c$ and consider the set of sentences

$$\Gamma = \text{TA} \cup \{c \neq 0, c \neq 1, c \neq 2, \ldots \}$$

Any model $\mathfrak{M}^c$ of $\Gamma$ would contain an element $x = c^\mathfrak{M}$ which is non-standard, since $x \neq \text{Val}^\mathfrak{M}(\pi)$ for all $n \in \mathbb{N}$. Also, obviously, $\mathfrak{M}^c \models \text{TA}$, since $\text{TA} \subseteq \Gamma$. If we turn $\mathfrak{M}^c$ into a structure $\mathfrak{M}$ for $\mathcal{L}_A$ simply by forgetting about $c$, its domain still contains the non-standard $x$, and also $\mathfrak{M} \models \text{TA}$. The latter is guaranteed since $c$ does not occur in $\text{TA}$. So, it suffices to show that $\Gamma$ has a model.

We use the compactness theorem to show that $\Gamma$ has a model. If every finite subset of $\Gamma$ is satisfiable, so is $\Gamma$. Consider any finite subset $\Gamma_0 \subseteq \Gamma$. $\Gamma_0$ includes some sentences of $\text{TA}$ and some of the form $c \neq \pi$, but only finitely many. Suppose $k$ is the largest number so that $c \neq k \in \Gamma_0$. Define $\mathfrak{M}_k$ by expanding $\mathfrak{M}$ to include the interpretation $c^\mathfrak{M}_k = k + 1$. $\mathfrak{M}_k \models \Gamma_0$: if $\varphi \in \text{TA}$, $\mathfrak{M}_k \models \varphi$ since $\mathfrak{M}_k$ is just like $\mathfrak{M}$ in all respects except $c$, and $c$ does not occur in $\varphi$. And $\mathfrak{M}_k \models c \neq \pi$, since $n \leq k$, and $\text{Val}^\mathfrak{M}_k(c) = k + 1$. Thus, every finite subset of $\Gamma$ is satisfiable. \hfill \Box

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Bibliography