

mar.1 Non-Standard Models

We call a **structure** for \mathcal{L}_A standard if it is isomorphic to \mathfrak{N} . If a **structure** isn't isomorphic to \mathfrak{N} , it is called non-standard. explanation

Definition mar.1. A **structure** \mathfrak{M} for \mathcal{L}_A is *non-standard* if it is not isomorphic to \mathfrak{N} . The **elements** $x \in |\mathfrak{M}|$ which are equal to $\text{Val}^{\mathfrak{M}}(\bar{n})$ for some $n \in \mathbb{N}$ are called *standard numbers* (of \mathfrak{M}), and those not, *non-standard numbers*.

By ??, any standard **structure** for \mathcal{L}_A contains only standard **elements**. explanation Consequently, a non-standard **structure** must contain at least one non-standard element. In fact, the existence of a non-standard **element** guarantees that the **structure** is non-standard.

Proposition mar.2. *If a structure \mathfrak{M} for \mathcal{L}_A contains a non-standard number, \mathfrak{M} is non-standard.*

Proof. Suppose not, i.e., suppose \mathfrak{M} standard but contains a non-standard number x . Let $g: \mathbb{N} \rightarrow |\mathfrak{M}|$ be an isomorphism. It is easy to see (by induction on n) that $g(\text{Val}^{\mathfrak{N}}(\bar{n})) = \text{Val}^{\mathfrak{M}}(\bar{n})$. In other words, g maps standard numbers of \mathfrak{N} to standard numbers of \mathfrak{M} . If \mathfrak{M} contains a non-standard number, g cannot be **surjective**, contrary to hypothesis. \square

Problem mar.1. Recall that **Q** contains the axioms

$$\forall x \forall y (x' = y' \rightarrow x = y) \quad (Q_1)$$

$$\forall x \ 0 \neq x' \quad (Q_2)$$

$$\forall x (x = 0 \vee \exists y x = y') \quad (Q_3)$$

Give **structures** $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$ such that

1. $\mathfrak{M}_1 \models Q_1, \mathfrak{M}_1 \models Q_2, \mathfrak{M}_1 \not\models Q_3$;
2. $\mathfrak{M}_2 \models Q_1, \mathfrak{M}_2 \not\models Q_2, \mathfrak{M}_2 \models Q_3$; and
3. $\mathfrak{M}_3 \not\models Q_1, \mathfrak{M}_3 \models Q_2, \mathfrak{M}_3 \models Q_3$;

Obviously, you just have to specify $0^{\mathfrak{M}_i}$ and $\prime^{\mathfrak{M}_i}$ for each.

It is easy enough to specify non-standard **structures** for \mathcal{L}_A . For instance, explanation take the structure with **domain** \mathbb{Z} and interpret all non-logical symbols as usual. Since negative numbers are not values of \bar{n} for any n , this structure is non-standard. Of course, it will not be a *model* of arithmetic in the sense that it makes the same sentences true as \mathfrak{N} . For instance, $\forall x x' \neq 0$ is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

Proposition mar.3. *Let $\mathbf{TA} = \{\varphi : \mathfrak{N} \models \varphi\}$ be the theory of \mathfrak{N} . \mathbf{TA} has an **enumerable non-standard model**.*

Proof. Expand \mathcal{L}_A by a new constant symbol c and consider the set of sentences

$$\Gamma = \mathbf{TA} \cup \{c \neq \bar{0}, c \neq \bar{1}, c \neq \bar{2}, \dots\}$$

Any model \mathfrak{M}^c of Γ would contain an element $x = c^{\mathfrak{M}}$ which is non-standard, since $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$ for all $n \in \mathbb{N}$. Also, obviously, $\mathfrak{M}^c \models \mathbf{TA}$, since $\mathbf{TA} \subseteq \Gamma$. If we turn \mathfrak{M}^c into a structure \mathfrak{M} for \mathcal{L}_A simply by forgetting about c , its domain still contains the non-standard x , and also $\mathfrak{M} \models \mathbf{TA}$. The latter is guaranteed since c does not occur in \mathbf{TA} . So, it suffices to show that Γ has a model.

We use the compactness theorem to show that Γ has a model. If every finite subset of Γ is satisfiable, so is Γ . Consider any finite subset $\Gamma_0 \subseteq \Gamma$. Γ_0 includes some sentences of \mathbf{TA} and some of the form $c \neq \bar{n}$, but only finitely many. Suppose k is the largest number so that $c \neq \bar{k} \in \Gamma_0$. Define \mathfrak{N}_k by expanding \mathfrak{N} to include the interpretation $c^{\mathfrak{N}_k} = k + 1$. $\mathfrak{N}_k \models \Gamma_0$: if $\varphi \in \mathbf{TA}$, $\mathfrak{N}_k \models \varphi$ since \mathfrak{N}_k is just like \mathfrak{N} in all respects except c , and c does not occur in φ . And $\mathfrak{N}_k \models c \neq \bar{n}$, since $n \leq k$, and $\text{Val}^{\mathfrak{N}_k}(c) = k + 1$. Thus, every finite subset of Γ is satisfiable. \square

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Bibliography