mar.1 Non-Standard Models

We call a structure for \( \mathcal{L}_A \) standard if it is isomorphic to \( \mathfrak{N} \). If a structure isn’t isomorphic to \( \mathfrak{N} \), it is called non-standard.

**Definition mar.1.** A structure \( \mathcal{M} \) for \( \mathcal{L}_A \) is non-standard if it is not isomorphic to \( \mathfrak{N} \). The elements \( x \in |\mathcal{M}| \) which are equal to \( \text{Val}^{\mathcal{M}}(\pi) \) for some \( n \in \mathbb{N} \) are called standard numbers (of \( \mathcal{M} \)), and those not, non-standard numbers.

By ??, any standard structure for \( \mathcal{L}_A \) contains only standard elements. Consequently, a non-standard structure must contain at least one non-standard element. In fact, the existence of a non-standard element guarantees that the structure is non-standard.

**Proposition mar.2.** If a structure \( \mathcal{M} \) for \( \mathcal{L}_A \) contains a non-standard number, \( \mathcal{M} \) is non-standard.

**Proof.** Suppose not, i.e., suppose \( \mathcal{M} \) standard but contains a non-standard number \( x \). Let \( g: \mathbb{N} \to |\mathcal{M}| \) be an isomorphism. It is easy to see (by induction on \( n \)) that \( g(\text{Val}^{\mathfrak{N}}(\pi)) = \text{Val}^{\mathcal{M}}(\pi) \). In other words, \( g \) maps standard numbers of \( \mathfrak{N} \) to standard numbers of \( \mathcal{M} \). If \( \mathcal{M} \) contains a non-standard number, \( g \) cannot be surjective, contrary to hypothesis.

**Problem mar.1.** Recall that \( \mathcal{Q} \) contains the axioms

\[
\begin{align*}
\forall x \forall y (x’ = y’ &\rightarrow x = y) & (Q_1) \\
\forall x x’ \neq x & (Q_2) \\
\forall x (x \neq 0 &\rightarrow \exists y x = y’) & (Q_3)
\end{align*}
\]

Give structures \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \) such that

1. \( \mathcal{M}_1 \models Q_1, \mathcal{M}_1 \models Q_2, \mathcal{M}_1 \not\models Q_3; \)
2. \( \mathcal{M}_2 \models Q_1, \mathcal{M}_2 \not\models Q_2, \mathcal{M}_2 \models Q_3; \) and
3. \( \mathcal{M}_3 \not\models Q_1, \mathcal{M}_3 \models Q_2, \mathcal{M}_3 \models Q_3; \)

Obviously, you just have to specify \( o^{\mathcal{M}_1} \) and \( o^{\mathcal{M}_2} \) for each.

It is easy enough to specify non-standard structures for \( \mathcal{L}_A \). For instance, take the structure with domain \( \mathbb{Z} \) and interpret all non-logical symbols as usual. Since negative numbers are not values of \( \pi \) for any \( n \), this structure is non-standard. Of course, it will not be a model of arithmetic in the sense that it makes the same sentences true as \( \mathfrak{N} \). For instance, \( \forall x x’ \neq 0 \) is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

**Proposition mar.3.** Let \( \mathcal{T}_A = \{ \varphi : \mathfrak{N} \models \varphi \} \) be the theory of \( \mathfrak{N} \). \( \mathcal{T}_A \) has an enumerable non-standard model.
Proof. Expand $\mathcal{L}_A$ by a new constant symbol $c$ and consider the set of sentences

$$\Gamma = \mathbf{TA} \cup \{ c \neq 0, c \neq 1, c \neq 2, \ldots \}$$

Any model $\mathcal{M}^c$ of $\Gamma$ would contain an element $x = c^{\mathcal{M}}$ which is non-standard, since $x \neq \text{Val}^{\mathcal{M}}(n)$ for all $n \in \mathbb{N}$. Also, obviously, $\mathcal{M}^c \models \mathbf{TA}$, since $\mathbf{TA} \subseteq \Gamma$. If we turn $\mathcal{M}^c$ into a structure $\mathcal{M}$ for $\mathcal{L}_A$ simply by forgetting about $c$, its domain still contains the non-standard $x$, and also $\mathcal{M} \models \mathbf{TA}$. The latter is guaranteed since $c$ does not occur in $\mathbf{TA}$. So, it suffices to show that $\Gamma$ has a model.

We use the compactness theorem to show that $\Gamma$ has a model. If every finite subset of $\Gamma$ is satisfiable, so is $\Gamma$. Consider any finite subset $\Gamma_0 \subseteq \Gamma$. $\Gamma_0$ includes some sentences of $\mathbf{TA}$ and some of the form $c \neq n$, but only finitely many. Suppose $k$ is the largest number so that $c \neq k \in \Gamma_0$. Define $\mathcal{M}_k$ by expanding $\mathcal{M}$ to include the interpretation $c^{\mathcal{M}_k} = k + 1$. $\mathcal{M}_k \models \Gamma_0$: if $\varphi \in \mathbf{TA}$, $\mathcal{M}_k \models \varphi$ since $\mathcal{M}_k$ is just like $\mathcal{M}$ in all respects except $c$, and $c$ does not occur in $\varphi$. And $\mathcal{M}_k \models c \neq n$, since $n \leq k$, and $\text{Val}^{\mathcal{M}_k}(c) = k + 1$. Thus, every finite subset of $\Gamma$ is satisfiable. \qed

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Bibliography