mar.1 Non-Standard Models

We call a structure for $\mathcal{L}_A$ standard if it is isomorphic to $\mathcal{N}$. If a structure isn’t isomorphic to $\mathcal{N}$, it is called non-standard.

**Definition mar.1.** A structure $\mathcal{M}$ for $\mathcal{L}_A$ is non-standard if it is not isomorphic to $\mathcal{N}$. The elements $x \in |\mathcal{M}|$ which are equal to $\text{Val}_\mathcal{M}(\pi)$ for some $n \in \mathbb{N}$ are called standard numbers (of $\mathcal{M}$), and those not, non-standard numbers.

By ??, any standard structure for $\mathcal{L}_A$ contains only standard elements. Consequently, a non-standard structure must contain at least one non-standard element. In fact, the existence of a non-standard element guarantees that the structure is non-standard.

**Proposition mar.2.** If a structure $\mathcal{M}$ for $\mathcal{L}_A$ contains a non-standard number, $\mathcal{M}$ is non-standard.

**Proof.** Suppose not, i.e., suppose $\mathcal{M}$ standard but contains a non-standard number $x$. Let $g: \mathbb{N} \to |\mathcal{M}|$ be an isomorphism. It is easy to see (by induction on $n$) that $g(\text{Val}_\mathcal{N}(\pi)) = \text{Val}_\mathcal{M}(\pi)$. In other words, $g$ maps standard numbers of $\mathcal{N}$ to standard numbers of $\mathcal{M}$. If $\mathcal{M}$ contains a non-standard number, $g$ cannot be surjective, contrary to hypothesis.

**Problem mar.1.** Recall that $\mathcal{Q}$ contains the axioms

\[
\forall x \forall y (x' = y' \to x = y) \quad (Q_1)
\]
\[
\forall x x \neq x' \quad (Q_2)
\]
\[
\forall x (x = 0 \lor \exists y x = y') \quad (Q_3)
\]

Give structures $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ such that

1. $\mathcal{M}_1 \models Q_1$, $\mathcal{M}_1 \not\models Q_2$, $\mathcal{M}_1 \not\models Q_3$;
2. $\mathcal{M}_2 \models Q_1$, $\mathcal{M}_2 \not\models Q_2$, $\mathcal{M}_2 \models Q_3$; and
3. $\mathcal{M}_3 \not\models Q_1$, $\mathcal{M}_3 \models Q_2$, $\mathcal{M}_3 \models Q_3$;

Obviously, you just have to specify $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, for each.

It is easy enough to specify non-standard structures for $\mathcal{L}_A$. For instance, take the structure with domain $\mathbb{Z}$ and interpret all non-logical symbols as usual. Since negative numbers are not values of $\pi$ for any $n$, this structure is non-standard. Of course, it will not be a model of arithmetic in the sense that it makes the same sentences true as $\mathcal{N}$. For instance, $\forall x x' \neq 0$ is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

**Proposition mar.3.** Let $\mathcal{T}A = \{ \varphi : \mathcal{N} \models \varphi \}$ be the theory of $\mathcal{N}$. $\mathcal{T}A$ has an enumerable non-standard model.
Proof. Expand $\mathcal{L}_A$ by a new constant symbol $c$ and consider the set of sentences

$$\Gamma = \text{TA} \cup \{c \neq 0, c \neq 1, c \neq 2, \ldots \}$$

Any model $\mathfrak{M}^c$ of $\Gamma$ would contain an element $x = c^n_{\mathfrak{M}}$ which is non-standard, since $x \neq \text{Val}^\mathfrak{M}(\pi)$ for all $n \in \mathbb{N}$. Also, obviously, $\mathfrak{M}^c \models \text{TA}$, since $\text{TA} \subseteq \Gamma$. If we turn $\mathfrak{M}^c$ into a structure $\mathfrak{M}$ for $\mathcal{L}_A$ simply by forgetting about $c$, its domain still contains the non-standard $x$, and also $\mathfrak{M} \models \text{TA}$. The latter is guaranteed since $c$ does not occur in $\text{TA}$. So, it suffices to show that $\Gamma$ has a model.

We use the compactness theorem to show that $\Gamma$ has a model. If every finite subset of $\Gamma$ is satisfiable, so is $\Gamma$. Consider any finite subset $\Gamma_0 \subseteq \Gamma$. $\Gamma_0$ includes some sentences of $\text{TA}$ and some of the form $c \neq n$, but only finitely many. Suppose $k$ is the largest number so that $c \neq k \in \Gamma_0$. Define $\mathfrak{M}_k$ by expanding $\mathfrak{M}$ to include the interpretation $c^\mathfrak{M}_k = k + 1$. $\mathfrak{M}_k \models \Gamma_0$: if $\varphi \in \text{TA}$, $\mathfrak{M}_k \models \varphi$ since $\mathfrak{M}_k$ is just like $\mathfrak{M}$ in all respects except $c$, and $c$ does not occur in $\varphi$. And $\mathfrak{M}_k \models c \neq n$, since $n \leq k$, and $\text{Val}^{\mathfrak{M}_k}(c) = k + 1$. Thus, every finite subset of $\Gamma$ is satisfiable. 

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Bibliography