

## mar.1 Models of $\mathbf{Q}$

We know that there are non-standard **structures** that make the same **sentences** true as  $\mathfrak{N}$  does, i.e., is a model of **TA**. Since  $\mathfrak{N} \models \mathbf{Q}$ , any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g.,  $\mathbf{Q} \not\models \forall x \forall y (x+y) = (y+x)$ . Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make  $\forall x \forall y (x+y) = (y+x)$  false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly.

mod:mar:mdq:  
ex:model-K-of-Q

**Example mar.1.** Consider the **structure**  $\mathfrak{K}$  with domain  $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$  and interpretations

$$\begin{aligned} 0^{\mathfrak{K}} &= 0 \\ \iota^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x+y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

To show that  $\mathfrak{K} \models \mathbf{Q}$  we have to verify that all axioms of **Q** are true in  $\mathfrak{K}$ . For convenience, let's write  $x^*$  for  $\iota^{\mathfrak{K}}(x)$  (the “successor” of  $x$  in  $\mathfrak{K}$ ),  $x \oplus y$  for  $+^{\mathfrak{K}}(x, y)$  (the “sum” of  $x$  and  $y$  in  $\mathfrak{K}$ ),  $x \otimes y$  for  $\times^{\mathfrak{K}}(x, y)$  (the “product” of  $x$  and  $y$  in  $\mathfrak{K}$ ), and  $x \ominus y$  for  $\langle x, y \rangle \in <^{\mathfrak{K}}$ . With these abbreviations, we can give the operations in  $\mathfrak{K}$  more perspicuously as

$x$	$x^*$	$x \oplus y$	$m$	$a$	$x \otimes y$	$m$	$a$
$n$	$n+1$	$n$	$n+m$	$a$	$n$	$nm$	$a$
$a$	$a$	$a$	$a$	$a$	$a$	$a$	$a$

We have  $n \ominus m$  iff  $n < m$  for  $n, m \in \mathbb{N}$  and  $x \ominus a$  for all  $x \in |\mathfrak{K}|$ .

$\mathfrak{K} \models \forall x \forall y (x' = y' \rightarrow x = y)$  since  $*$  is **injective**.  $\mathfrak{K} \models \forall x 0 \neq x'$  since 0 is not a  $*$ -successor in  $\mathfrak{K}$ .  $\mathfrak{N} \models \forall x (x \neq 0 \rightarrow \exists y x = y')$  since for every  $n > 0$ ,  $n = (n-1)^*$ , and  $a = a^*$ .

$\mathfrak{K} \models \forall x (x \oplus 0) = x$  since  $n \oplus 0 = n + 0 = n$ , and  $a \oplus 0 = a$  by definition of  $\oplus$ .  $\mathfrak{K} \models \forall x \forall y (x \oplus y') = (x+y)'$  is a bit trickier. If  $n, m$  are both standard, we have:

$$(n \oplus m^*) = (n + (m+1)) = (n+m) + 1 = (n \oplus m)^*$$

since  $\oplus$  and  $*$  agree with  $+$  and  $\iota$  on standard numbers. Now suppose  $x \in |\mathfrak{K}|$ . Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if  $y \in |\mathfrak{K}|$  but  $x = a$ . Here we also have to distinguish cases according to whether  $y = n$  is standard or  $y = b$ :

$$\begin{aligned}(a \oplus n^*) &= (a \oplus (n + 1)) = a = a^* = (x \oplus n)^* \\ (a \oplus a^*) &= (a \oplus a) = a = a^* = (x \oplus a)^*\end{aligned}$$

This is of course a bit more detailed than needed. For instance, since  $a \oplus z = a$  whatever  $z$  is, we can immediately conclude  $a \oplus a^* = a$ . The remaining axioms can be verified the same way.

$\mathfrak{K}$  is thus a model of  $\mathbf{Q}$ . Its “addition”  $\oplus$  is also commutative. But there are other sentences true in  $\mathfrak{N}$  but false in  $\mathfrak{K}$ , and vice versa. For instance,  $a \otimes a$ , so  $\mathfrak{K} \models \exists x x < x$  and  $\mathfrak{K} \not\models \forall x \neg x < x$ . This shows that  $\mathbf{Q} \not\models \forall x \neg x < x$ .

**Problem mar.1.** Prove that  $\mathfrak{K}$  from [Example mar.1](#) satisfies the remaining axioms of  $\mathbf{Q}$ ,

$$\forall x (x \times 0) = 0 \tag{Q6}$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \tag{Q7}$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (x + z' = y)) \tag{Q8}$$

Find a [sentence](#) only involving  $\prime$  true in  $\mathfrak{N}$  but false in  $\mathfrak{K}$ .

**Example mar.2.** Consider the [structure](#)  $\mathfrak{L}$  with domain  $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$  and interpretations  $\prime^{\mathfrak{L}} = *$ ,  $+^{\mathfrak{L}} = \oplus$  given by

[mod:mar:mdq:](#)  
[ex:model-L-of-Q](#)

$x$	$x^*$	$x \oplus y$	$m$	$a$	$b$
$n$	$n + 1$	$n$	$n + m$	$b$	$a$
$a$	$a$	$a$	$a$	$b$	$a$
$b$	$b$	$b$	$b$	$b$	$a$

Since  $*$  is [injective](#),  $0$  is not in its range, and every  $x \in |\mathfrak{L}|$  other than  $0$  is, axioms  $Q_1$ – $Q_3$  are true in  $\mathfrak{L}$ . For any  $x$ ,  $x \oplus 0 = x$ , so  $Q_4$  is true as well. For  $Q_5$ , consider  $x \oplus y^*$  and  $(x \oplus y)^*$ . They are equal if  $x$  and  $y$  are both standard, since then  $*$  and  $\oplus$  agree with  $\prime$  and  $+$ . If  $x$  is non-standard, and  $y$  is standard, we have  $x \oplus y^* = x = x^* = (x \oplus y)^*$ . If  $x$  and  $y$  are both non-standard, we have four cases:

$$\begin{aligned}a \oplus a^* &= b = b^* = (a \oplus a)^* \\ b \oplus b^* &= a = a^* = (b \oplus b)^* \\ b \oplus a^* &= b = b^* = (b \oplus y)^* \\ a \oplus b^* &= a = a^* = (a \oplus b)^*\end{aligned}$$

If  $x$  is standard, but  $y$  is non-standard, we have

$$\begin{aligned}n \oplus a^* &= n \oplus a = b = b^* = (n \oplus a)^* \\ n \oplus b^* &= n \oplus b = a = a^* = (n \oplus b)^*\end{aligned}$$

So,  $\mathfrak{L} \models Q_5$ . However,  $a \oplus 0 \neq 0 \oplus a$ , so  $\mathfrak{L} \not\models \forall x \forall y (x + y) = (y + x)$ .

**Problem mar.2.** Expand  $\mathcal{L}$  of [Example mar.2](#) to include  $\otimes$  and  $\oslash$  that interpret  $\times$  and  $<$ . Show that your structure satisfies the remaining axioms of  $\mathbf{Q}$ ,

$$\forall x (x \times 0) = 0 \quad (Q_6)$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \quad (Q_7)$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (x + z' = y)) \quad (Q_8)$$

**Problem mar.3.** In  $\mathcal{L}$  of [Example mar.2](#),  $a^* = a$  and  $b^* = b$ . Is there a model of  $\mathbf{Q}$  in which  $a^* = b$  and  $b^* = a$ ?

We've explicitly constructed models of  $\mathbf{Q}$  in which the non-standard [elements](#) live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard [element](#)  $x$  cannot be  $\oslash 0$ . Otherwise, for some  $z$ ,  $x \oplus z^* = 0$  by  $Q_8$ . But then  $0 = x \oplus z^* = (x \oplus z)^*$  by  $Q_5$ , contradicting  $Q_2$ . Also, for every  $n$ ,  $\mathbf{Q} \vdash \forall x (x < \bar{n}' \rightarrow (x = \bar{0} \vee x = \bar{1} \vee \dots \vee x = \bar{n}))$ , so we can't have  $a \oslash n$  for any  $n > 0$ .

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## Bibliography