

mar.1 Models of \mathbf{Q}

We know that there are non-standard **structures** that make the same **sentences** true as \mathfrak{N} does, i.e., is a model of **TA**. Since $\mathfrak{N} \models \mathbf{Q}$, any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g., $\mathbf{Q} \not\models \forall x \forall y (x+y) = (y+x)$. Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make $\forall x \forall y (x+y) = (y+x)$ false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly.

mod:mar:mdq:
ex:model-K-of-Q

Example mar.1. Consider the **structure** \mathfrak{K} with domain $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$$\begin{aligned} 0^{\mathfrak{K}} &= 0 \\ \iota^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x+y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

To show that $\mathfrak{K} \models \mathbf{Q}$ we have to verify that all axioms of **Q** are true in \mathfrak{K} . For convenience, let's write x^* for $\iota^{\mathfrak{K}}(x)$ (the “successor” of x in \mathfrak{K}), $x \oplus y$ for $+^{\mathfrak{K}}(x, y)$ (the “sum” of x and y in \mathfrak{K}), $x \otimes y$ for $\times^{\mathfrak{K}}(x, y)$ (the “product” of x and y in \mathfrak{K}), and $x \odot y$ for $\langle x, y \rangle \in <^{\mathfrak{K}}$. With these abbreviations, we can give the operations in \mathfrak{K} more perspicuously as

x	x^*	$x \oplus y$	m	a	$x \otimes y$	m	a
n	$n+1$	n	$n+m$	a	n	nm	a
a	a	a	a	a	a	a	a

We have $n \odot m$ iff $n < m$ for $n, m \in \mathbb{N}$ and $x \odot a$ for all $x \in |\mathfrak{K}|$.

$\mathfrak{K} \models \forall x \forall y (x' = y' \rightarrow x = y)$ since $*$ is **injective**. $\mathfrak{K} \models \forall x 0 \neq x'$ since 0 is not a $*$ -successor in \mathfrak{K} . $\mathfrak{N} \models \forall x (x \neq 0 \rightarrow \exists y x = y')$ since for every $n > 0$, $n = (n-1)^*$, and $a = a^*$.

$\mathfrak{K} \models \forall x (x + 0) = x$ since $n \oplus 0 = n + 0 = n$, and $a \oplus 0 = a$ by definition of \oplus . $\mathfrak{K} \models \forall x \forall y (x + y') = (x + y)'$ is a bit trickier. If n, m are both standard, we have:

$$(n \oplus m^*) = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*$$

since \oplus and $*$ agree with $+$ and ι on standard numbers. Now suppose $x \in |\mathfrak{K}|$. Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if $y \in |\mathfrak{K}|$ but $x = a$. Here we also have to distinguish cases according to whether $y = n$ is standard or $y = b$:

$$\begin{aligned}(a \oplus n^*) &= (a \oplus (n + 1)) = a = a^* = (x \oplus n)^* \\ (a \oplus a^*) &= (a \oplus a) = a = a^* = (x \oplus a)^*\end{aligned}$$

This is of course a bit more detailed than needed. For instance, since $a \oplus z = a$ whatever z is, we can immediately conclude $a \oplus a^* = a$. The remaining axioms can be verified the same way.

\mathfrak{K} is thus a model of \mathbf{Q} . Its “addition” \oplus is also commutative. But there are other sentences true in \mathfrak{N} but false in \mathfrak{K} , and vice versa. For instance, $a \otimes a$, so $\mathfrak{K} \models \exists x x < x$ and $\mathfrak{K} \not\models \forall x \neg x < x$. This shows that $\mathbf{Q} \not\models \forall x \neg x < x$.

Problem mar.1. Prove that \mathfrak{K} from [Example mar.1](#) satisfies the remaining axioms of \mathbf{Q} ,

$$\forall x (x \times 0) = 0 \tag{Q6}$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \tag{Q7}$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (x + z' = y)) \tag{Q8}$$

Find a [sentence](#) only involving \prime true in \mathfrak{N} but false in \mathfrak{K} .

Example mar.2. Consider the [structure](#) \mathfrak{L} with domain $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$ and interpretations $\prime^{\mathfrak{L}} = *$, $+^{\mathfrak{L}} = \oplus$ given by

[mod:mar:mdq:](#)
[ex:model-L-of-Q](#)

x	x^*	$x \oplus y$	m	a	b
n	$n + 1$	n	$n + m$	b	a
a	a	a	a	b	a
b	b	b	b	b	a

Since $*$ is [injective](#), 0 is not in its range, and every $x \in |\mathfrak{L}|$ other than 0 is, axioms Q_1 – Q_3 are true in \mathfrak{L} . For any x , $x \oplus 0 = x$, so Q_4 is true as well. For Q_5 , consider $x \oplus y^*$ and $(x \oplus y)^*$. They are equal if x and y are both standard, since then $*$ and \oplus agree with \prime and $+$. If x is non-standard, and y is standard, we have $x \oplus y^* = x = x^* = (x \oplus y)^*$. If x and y are both non-standard, we have four cases:

$$\begin{aligned}a \oplus a^* &= b = b^* = (a \oplus a)^* \\ b \oplus b^* &= a = a^* = (b \oplus b)^* \\ b \oplus a^* &= b = b^* = (b \oplus y)^* \\ a \oplus b^* &= a = a^* = (a \oplus b)^*\end{aligned}$$

If x is standard, but y is non-standard, we have

$$\begin{aligned}n \oplus a^* &= n \oplus a = b = b^* = (n \oplus a)^* \\ n \oplus b^* &= n \oplus b = a = a^* = (n \oplus b)^*\end{aligned}$$

So, $\mathfrak{L} \models Q_5$. However, $a \oplus 0 \neq 0 \oplus a$, so $\mathfrak{L} \not\models \forall x \forall y (x + y) = (y + x)$.

Problem mar.2. Expand \mathcal{L} of [Example mar.2](#) to include \otimes and \oslash that interpret \times and $<$. Show that your structure satisfies the remaining axioms of \mathbf{Q} ,

$$\forall x (x \times 0) = 0 \quad (Q_6)$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \quad (Q_7)$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (x + z' = y)) \quad (Q_8)$$

Problem mar.3. In \mathcal{L} of [Example mar.2](#), $a^* = a$ and $b^* = b$. Is there a model of \mathbf{Q} in which $a^* = b$ and $b^* = a$?

We've explicitly constructed models of \mathbf{Q} in which the non-standard [elements](#) live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard [element](#) x cannot be $\oslash 0$. Otherwise, for some z , $x \oplus z^* = 0$ by Q_8 . But then $0 = x \oplus z^* = (x \oplus z)^*$ by Q_5 , contradicting Q_2 . Also, for every n , $\mathbf{Q} \vdash \forall x (x < \bar{n}' \rightarrow (x = \bar{0} \vee x = \bar{1} \vee \dots \vee x = \bar{n}))$, so we can't have $a \oslash n$ for any $n > 0$.

Photo Credits

Bibliography