mar.1 Models of Q

We know that there are non-standard structures that make the same sentences true as $\mathfrak{N}$ does, i.e., is a model of $\mathbf{TA}$. Since $\mathfrak{N} \models Q$, any model of $\mathbf{TA}$ is also a model of $Q$. $Q$ is much weaker than $\mathbf{TA}$, e.g., $Q \not\models \forall x \forall y (x + y) = (y + x)$. Weaker theories are easier to satisfy: they have more models. E.g., $Q$ has models which make $\forall x \forall y (x + y) = (y + x)$ false, but those cannot also be models of $\mathbf{TA}$, or $\mathbf{PA}$ for that matter. Models of $Q$ are also relatively simple: we can specify them explicitly.

Example mar.1. Consider the structure $\mathfrak{R}$ with domain $|\mathfrak{R}| = \mathbb{N} \cup \{a\}$ and interpretations

$$\sigma^\mathfrak{R} = 0$$
$$\rho^\mathfrak{R}(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases}$$
$$\star^\mathfrak{R}(x, y) = \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$$
$$\times^\mathfrak{R}(x, y) = \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$$

$$<^\mathfrak{R} = \{ (x, y) : x, y \in \mathbb{N} \text{ and } x < y \} \cup \{ (x, a) : x \in |\mathfrak{R}| \}$$

To show that $\mathfrak{R} \models Q$ we have to verify that all axioms of $Q$ are true in $\mathfrak{R}$. For convenience, let’s write $x^*$ for $\rho^\mathfrak{R}(x)$ (the “successor” of $x$ in $\mathfrak{R}$), $x \oplus y$ for $\star^\mathfrak{R}(x, y)$ (the “sum” of $x$ and $y$ in $\mathfrak{R}$), $x \odot y$ for $\times^\mathfrak{R}(x, y)$ (the “product” of $x$ and $y$ in $\mathfrak{R}$), and $x \bowtie y$ for $(x, y) \in <^\mathfrak{R}$. With these abbreviations, we can give the operations in $\mathfrak{R}$ more perspicuously as

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^*$</th>
<th>$x \odot y$</th>
<th>$x \bowtie y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$n + 1$</td>
<td>$n$</td>
<td>$m$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
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We have $n \odot m$ iff $n < m$ for $n, m \in \mathbb{N}$ and $x \bowtie a$ for all $x \in |\mathfrak{R}|$.

$\mathfrak{R} \models \forall x \forall y (x^* = y^* \rightarrow x = y)$ since $^*$ is injective. $\mathfrak{R} \models \forall x \odot y \neq x^*$ since $0$ is not a $^*$-successor in $\mathfrak{R}$. $\mathfrak{R} \models \forall x (x \neq 0 \rightarrow \exists y x = y')$ since for every $n > 0$, $n = (n - 1)^*$, and $a = a^*$.

$\mathfrak{R} \models \forall x (x \odot 0) = x$ since $n \odot 0 = n + 0 = n$, and $a \odot 0 = a$ by definition of $\odot$. $\mathfrak{R} \models \forall x \forall y (x + y^*) = (x + y)'$ is a bit trickier. If $n, m$ are both standard, we have:

$$(n \odot m^*) = (n + (m + 1)) = (n + m) + 1 = (n \odot m)^*$$

since $\odot$ and $^*$ agree with $+$ and $\star$ on standard numbers. Now suppose $x \in |\mathfrak{R}|$.

Then

$$(x \odot a^*) = (x \odot a) = a = a^* = (x \odot a)^*$$
The remaining case is if \( y \in |\mathcal{R}| \) but \( x = a \). Here we also have to distinguish cases according to whether \( y = n \) is standard or \( y = b \):

\[
(a \oplus n^*) = (a \oplus (n + 1)) = a = a^* = (x \oplus n)^*
\]

\[
(a \oplus a^*) = (a \oplus a) = a = a^* = (x \oplus a)^*
\]

This is of course a bit more detailed than needed. For instance, since \( a \oplus z = a \) whatever \( z \) is, we can immediately conclude \( a \oplus a^* = a \). The remaining axioms can be verified the same way.

\( \mathcal{R} \) is thus a model of \( Q \). Its “addition” \( \oplus \) is also commutative. But there are other sentences true in \( \mathcal{R} \) but false in \( \mathcal{R} \), and vice versa. For instance, \( a \oplus a \), so \( \mathcal{R} \models \exists x \ x < x \) and \( \mathcal{R} \not\models \forall x \neg x < x \). This shows that \( Q \not\models \forall \neg x < x \).

**Problem mar.1.** Prove that \( \mathcal{R} \) from Example mar.1 satisfies the remaining axioms of \( Q \):

\[
\forall x \ (x \times 0) = 0 \quad (Q_6)
\]
\[
\forall x \forall y \ (x \times y' = ((x \times y) + x) \quad (Q_7)
\]
\[
\forall x \forall y \ (x < y \iff \exists z \ (x + z' = y)) \quad (Q_8)
\]

Find a sentence only involving \( \tau \) true in \( \mathcal{R} \) but false in \( \mathcal{R} \).

**Example mar.2.** Consider the structure \( \mathcal{L} \) with domain \( |\mathcal{L}| = \mathbb{N} \cup \{a, b\} \) and interpretations \( \rho^L = * \), \( + = \oplus \) given by

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<thead>
<tr>
<th></th>
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<th>( x \times y )</th>
<th>( m )</th>
<th>( a )</th>
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<tbody>
<tr>
<td>( n )</td>
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<td>( a )</td>
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<tr>
<td>( b )</td>
<td>( n + a )</td>
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Since \( * \) is injective, \( 0 \) is not in its range, and every \( x \in |\mathcal{L}| \) other than \( 0 \) is, axioms \( Q_1-Q_3 \) are true in \( \mathcal{L} \). For any \( x, x \oplus 0 = x \), so \( Q_4 \) is true as well. For \( Q_5 \), consider \( x \oplus y^* \) and \( (x \oplus y)^* \). They are equal if \( x \) and \( y \) are both standard, since then \( * \) and \( \oplus \) agree with \( \tau \) and \( \tau \). If \( x \) is non-standard, and \( y \) is standard, we have \( x \oplus y^* = x = x^* = (x \oplus y)^* \). If \( x \) and \( y \) are both non-standard, we have four cases:

\[
\begin{align*}
a \oplus a^* &= b = b^* = (a \oplus a)^* \\
b \oplus b^* &= a = a^* = (b \oplus b)^* \\
b \oplus a^* &= b = b^* = (b \oplus y)^* \\
a \oplus b^* &= a = a^* = (a \oplus b)^*
\end{align*}
\]

If \( x \) is standard, but \( y \) is non-standard, we have

\[
\begin{align*}
n \oplus a^* &= n \oplus a = b = b^* = (n \oplus a)^* \\
n \oplus b^* &= n \oplus b = a = a^* = (n \oplus b)^*
\end{align*}
\]

So, \( \mathcal{L} \models Q_5 \). However, \( a \oplus 0 \neq 0 \oplus a \), so \( \mathcal{L} \not\models \forall x \forall y \ (x + y) = (y + x) \).
**Problem mar.2.** Expand \( \mathcal{L} \) of Example mar.2 to include \( \otimes \) and \( \oplus \) that interpret \( \times \) and \( < \). Show that your structure satisfies the remaining axioms of \( Q \).

\[
\begin{align*}
\forall x (x \times 0 &= 0) & (Q_6) \\
\forall x \forall y (x \times y' &= ((x \times y) + x)) & (Q_7) \\
\forall x \forall y (x < y &\iff \exists z (x + z' = y)) & (Q_8)
\end{align*}
\]

**Problem mar.3.** In \( \mathcal{L} \) of Example mar.2, \( a^* = a \) and \( b^* = b \). Is there a model of \( Q \) in which \( a^* = b \) and \( b^* = a \)?

We’ve explicitly constructed models of \( Q \) in which the non-standard elements live “beyond” the standard elements. In fact, that much is required by the axioms. A non-standard element \( x \) cannot be \( \ominus 0 \). Otherwise, for some \( z \), \( x \oplus z^* = 0 \) by \( Q_8 \). But then \( 0 = x \oplus z^* = (x \oplus z)^* \) by \( Q_5 \), contradicting \( Q_2 \). Also, for every \( n \), \( Q \vdash \forall x (x < \pi' \rightarrow (x = \overline{0} \lor x = \overline{1} \lor \cdots \lor x = \overline{n})) \), so we can’t have \( a \ominus n \) for any \( n > 0 \).

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**Bibliography**