mar.1 Models of Q

We know that there are non-standard structures that make the same sentences true as $\mathfrak{N}$ does, i.e., is a model of **TA**. Since $\mathfrak{N} \vDash \mathbf{Q}$, any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g., $\mathbf{Q} \nvDash \forall x\, \forall y\, (x+y) = (y+x)$. Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make $\forall x\, \forall y\, (x+y) = (y+x)$ false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly. explanation

mod:mar:mdq: ex:model-K-of-Q **Example mar.1.** Consider the structure $\mathfrak{K}$ with domain $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$$\begin{aligned}
\mathsf{o}^{\mathfrak{K}} &= 0 \\
\prime^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\
+^{\mathfrak{K}}(x,y) &= \begin{cases} x+y & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
\times^{\mathfrak{K}}(x,y) &= \begin{cases} xy & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
<^{\mathfrak{K}} &= \{\langle x, y\rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a\rangle : x \in |\mathfrak{K}|\}
\end{aligned}$$

To show that $\mathfrak{K} \vDash \mathbf{Q}$ we have to verify that all axioms of **Q** are true in $\mathfrak{K}$. For convenience, let's write x^* for $\prime^{\mathfrak{K}}(x)$ (the "successor" of x in $\mathfrak{K}$), $x \oplus y$ for $+^{\mathfrak{K}}(x,y)$ (the "sum" of x and y in $\mathfrak{K}$, $x \otimes y$ for $\times^{\mathfrak{K}}(x,y)$ (the "product" of x and y in $\mathfrak{K}$), and $x \lessdot y$ for $\langle x, y\rangle \in <^{\mathfrak{K}}$. With these abbreviations, we can give the operations in $\mathfrak{K}$ more perspicuously as

x	x^*
n	$n+1$
a	a

$x \oplus y$	m	a
n	$n+m$	a
a	a	a

$x \otimes y$	m	a
n	nm	a
a	a	a

We have $n \lessdot m$ iff $n < m$ for $n, m \in \mathbb{N}$ and $x \lessdot a$ for all $x \in |\mathfrak{K}|$.

$\mathfrak{K} \vDash \forall x\, \forall y\, (x' = y' \to x = y)$ since $*$ is injective. $\mathfrak{K} \vDash \forall x\, \mathsf{o} \neq x'$ since 0 is not a $*$-successor in $\mathfrak{K}$. $\mathfrak{N} \vDash \forall x\, (x \neq \mathsf{o} \to \exists y\, x = y')$ since for every $n > 0$, $n = (n-1)^*$, and $a = a^*$.

$\mathfrak{K} \vDash \forall x\, (x + \mathsf{o}) = x$ since $n \oplus 0 = n + 0 = n$, and $a \oplus 0 = a$ by definition of $\oplus$. $\mathfrak{K} \vDash \forall x\, \forall y\, (x + y') = (x+y)'$ is a bit trickier. If n, m are both standard, we have:

$$(n \oplus m^*) = (n + (m+1)) = (n+m) + 1 = (n \oplus m)^*$$

since $\oplus$ and * agree with $+$ and $\prime$ on standard numbers. Now suppose $x \in |\mathfrak{K}|$. Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if $y \in |\mathfrak{K}|$ but $x = a$. Here we also have to distinguish cases according to whether $y = n$ is standard or $y = b$:

$$(a \oplus n^*) = (a \oplus (n+1)) = a = a^* = (x \oplus n)^*$$
$$(a \oplus a^*) = (a \oplus a) = a = a^* = (x \oplus a)^*$$

This is of course a bit more detailed than needed. For instance, since $a \oplus z = a$ whatever z is, we can immediately conclude $a \oplus a^* = a$. The remaining axioms can be verified the same way.

$\mathfrak{K}$ is thus a model of $\mathbf{Q}$. Its "addition" $\oplus$ is also commutative. But there are other sentences true in $\mathfrak{N}$ but false in $\mathfrak{K}$, and vice versa. For instance, $a \otimes a$, so $\mathfrak{K} \models \exists x\, x < x$ and $\mathfrak{K} \not\models \forall x\, \neg x < x$. This shows that $\mathbf{Q} \nvdash \forall x\, \neg x < x$.

Problem mar.1. Prove that $\mathfrak{K}$ from Example mar.1 satisifies the remaining axioms of $\mathbf{Q}$,

$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \qquad (Q_6)$$
$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \qquad (Q_7)$$
$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (x + z' = y)) \qquad (Q_8)$$

Find a sentence only involving $'$ true in $\mathfrak{N}$ but false in $\mathfrak{K}$.

Example mar.2. Consider the structure $\mathfrak{L}$ with domain $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$ and interpretations $'^{\mathfrak{L}} = *$, $+^{\mathfrak{L}} = \oplus$ given by mod:mar:mdq:ex:model-L-of-Q

x	x^*
n	$n+1$
a	a
b	b

$x \oplus y$	m	a	b
n	$n+m$	b	a
a	a	b	a
b	b	b	a

Since $*$ is injective, 0 is not in its range, and every $x \in |\mathfrak{L}|$ other than 0 is, axioms Q_1–Q_3 are true in $\mathfrak{L}$. For any x, $x \oplus 0 = x$, so Q_4 is true as well. For Q_5, consider $x \oplus y^*$ and $(x \oplus y)^*$. They are equal if x and y are both standard, since then $*$ and $\oplus$ agree with $'$ and $+$. If x is non-standard, and y is standard, we have $x \oplus y^* = x = x^* = (x \oplus y)^*$. If x and y are both non-standard, we have four cases:

$$a \oplus a^* = b = b^* = (a \oplus a)^*$$
$$b \oplus b^* = a = a^* = (b \oplus b)^*$$
$$b \oplus a^* = b = b^* = (b \oplus y)^*$$
$$a \oplus b^* = a = a^* = (a \oplus b)^*$$

If x is standard, but y is non-standard, we have

$$n \oplus a^* = n \oplus a = b = b^* = (n \oplus a)^*$$
$$n \oplus b^* = n \oplus b = a = a^* = (n \oplus b)^*$$

So, $\mathfrak{L} \models Q_5$. However, $a \oplus 0 \neq 0 \oplus a$, so $\mathfrak{L} \not\models \forall x\, \forall y\, (x + y) = (y + x)$.

Problem mar.2. Expand $\mathfrak{L}$ of Example mar.2 to include $\otimes$ and $\oslash$ that interpret $\times$ and $<$. Show that your structure satisifies the remaining axioms of $\mathbf{Q}$,

$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$

$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$

$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (x + z' = y)) \tag{Q_8}$$

Problem mar.3. In $\mathfrak{L}$ of Example mar.2, $a^* = a$ and $b^* = b$. Is there a model of $\mathbf{Q}$ in which $a^* = b$ and $b^* = a$?

We've explicitly constructed models of $\mathbf{Q}$ in which the non-standard elements live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard element x cannot be $\oslash 0$. Otherwise, for some z, $x \oplus z^* = 0$ by $Q8$. But then $0 = x \oplus z^* = (x \oplus z)^*$ by Q_5, contradicting Q_2. Also, for every n, $\mathbf{Q} \vdash \forall x\, (x < \overline{n}' \rightarrow (x = \overline{0} \lor x = \overline{1} \lor \cdots \lor x = \overline{n}))$, so we can't have $a \oslash n$ for any $n > 0$. explanation

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Bibliography