mar.1  Models of Q

We know that there are non-standard structures that make the same sentences true as \( \mathcal{R} \) does, i.e., is a model of TA. Since \( \mathcal{R} \models Q \), any model of TA is also a model of Q. Q is much weaker than TA, e.g., Q \( \not\models \forall x \forall y (x + y) = (y + x) \).

Weaker theories are easier to satisfy: they have more models. E.g., Q has models which make \( \forall x \forall y (x + y) = (y + x) \) false, but those cannot also be models of TA, or PA for that matter. Models of Q are also relatively simple: we can specify them explicitly.

Example mar.1. Consider the structure \( \mathcal{R} \) with domain \( |\mathcal{R}| = \mathbb{N} \cup \{a\} \) and interpretations

\[
\begin{align*}
\sigma^\mathcal{R} &= 0 \\
\ast^\mathcal{R}(x) &= \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\
        a & \text{if } x = a \end{cases} \\
\ast^\mathcal{R}(x, y) &= \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\
        a & \text{otherwise} \end{cases} \\
\times^\mathcal{R}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\
        0 & \text{if } x = 0 \text{ or } y = 0 \\
        a & \text{otherwise} \end{cases} \\
\langle^\mathcal{R} &= \{ (x, y) : x, y \in \mathbb{N} \text{ and } x < y \} \cup \{ (x, a) : x \in |\mathcal{R}| \} \\
\end{align*}
\]

To show that \( \mathcal{R} \models Q \) we have to verify that all axioms of Q are true in \( \mathcal{R} \).

For convenience, let’s write \( x^* \) for \( \sigma^\mathcal{R}(x) \) (the “successor” of \( x \) in \( \mathcal{R} \)), \( x \oplus y \) for \( +^\mathcal{R}(x, y) \) (the “sum” of \( x \) and \( y \) in \( \mathcal{R} \)), \( x \otimes y \) for \( \times^\mathcal{R}(x, y) \) (the “product” of \( x \) and \( y \) in \( \mathcal{R} \)), and \( x \triangleleft y \) for \( (x, y) \in \langle^\mathcal{R} \). With these abbreviations, we can give the operations in \( \mathcal{R} \) more perspicuously as

\[
\begin{array}{c|c|c|c|c}
x & x^* & x \oplus y & m & a \\
\hline
n & n + 1 & 0 & m & a \\
a & a & n & n + m & a \\
\end{array}
\quad
\begin{array}{c|c|c|c}
x \otimes y & m & a \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & nm & a \\
0 & a & a & a \\
a & 0 & a & a \\
\end{array}
\]

We have \( n \oplus m \) iff \( n < m \) for \( n, m \in \mathbb{N} \) and \( x \triangleleft a \) for all \( x \in |\mathcal{R}| \).

\( \mathcal{R} \models \forall x \forall y (x' = y' \rightarrow x = y) \) since \( \ast \) is injective. \( \mathcal{R} \models \forall x \ast \neq x' \) since 0 is not a \( \ast \)-successor in \( \mathcal{R} \). \( \mathcal{R} \models \forall x (x = \ast \lor \exists y x = y ') \) since for every \( n > 0 \), \( n = (n - 1)^* \), and \( a = a^* \).

\( \mathcal{R} \models \forall x (x + 0) = x \) since \( n \oplus 0 = n + 0 = n \), and \( a \oplus 0 = a \) by definition of \( \oplus \). \( \mathcal{R} \models \forall x \forall y (x + y') = (x + y)' \) is a bit trickier. If \( n, m \) are both standard, we have:

\[
(n \oplus m^*) = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*
\]

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since $\oplus$ and $\cdot$ agree with $+$ and $\prime$ on standard numbers. Now suppose $x \in |\mathcal{R}|$. Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if $y \in |\mathcal{R}|$ but $x = a$. Here we also have to distinguish cases according to whether $y = n$ is standard or $y = b$:

$$(a \oplus n^*) = (a \oplus (n + 1)) = a = a^* = (a \oplus n)^*$$

$$(a \oplus a^*) = (a \oplus a) = a = a^* = (a \oplus a)^*$$

This is of course a bit more detailed than needed. For instance, since $a \oplus z = a$ whatever $z$ is, we can immediately conclude $a \oplus a^* = a$. The remaining axioms can be verified the same way.

$\mathcal{R}$ is thus a model of $\mathbb{Q}$. Its “addition” $\oplus$ is also commutative. But there are other sentences true in $\mathbb{N}$ but false in $\mathcal{R}$, and vice versa. For instance, $a \oplus a$, so $\mathcal{R} \models \exists x (x < x)$ and $\mathcal{R} \not\models \forall x \lnot x < x$. This shows that $\mathbb{Q} \not\models \forall x \lnot x < x$.

**Problem mar.1.** Prove that $\mathcal{R}$ from Example mar.1 satisfies the remaining axioms of $\mathbb{Q}$,

$$\forall x (x \times 0) = 0 \quad (Q_6)$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \quad (Q_7)$$

$$\forall x \forall y (x < y \iff \exists z (z' + x) = y) \quad (Q_8)$$

Find a sentence only involving $\prime$ true in $\mathbb{N}$ but false in $\mathcal{R}$.

**Example mar.2.** Consider the structure $\mathcal{L}$ with domain $|\mathcal{L}| = \mathbb{N} \cup \{a, b\}$ and interpretations $\mathcal{L}^\mathcal{R} = \ast, +^{\mathcal{R}} = \oplus$ given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^*$</th>
<th>$x \oplus y$</th>
<th>$m$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$n + 1$</td>
<td>$n$</td>
<td>$n + m$</td>
<td>$b$</td>
<td>$a$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>$a$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>$a$</td>
</tr>
</tbody>
</table>

Since $\ast$ is injective, 0 is not in its range, and every $x \in |\mathcal{L}|$ other than 0 is, axioms $Q_1$–$Q_3$ are true in $\mathcal{L}$. For any $x$, $x \oplus 0 = x$, so $Q_4$ is true as well. For $Q_5$, consider $x \oplus y^*$ and $(x \oplus y)^*$. They are equal if $x$ and $y$ are both standard, since then $\ast$ and $\oplus$ agree with $\prime$ and $\prime$. If $x$ is non-standard, and $y$ is standard, we have $x \oplus y^* = x = x^* = (x \oplus y)^*$. If $x$ and $y$ are both non-standard, we have four cases:

$$a \oplus a^* = b = b^* = (a \oplus a)^*$$

$$b \oplus b^* = a = a^* = (b \oplus b)^*$$

$$b \oplus a^* = b = b^* = (b \oplus y)^*$$

$$a \oplus b^* = a = a^* = (a \oplus b)^*$$
If \( x \) is standard, but \( y \) is non-standard, we have

\[
\begin{align*}
n \oplus a^* &= n \oplus a = b = b^* = (n \oplus a)^* \\
n \oplus b^* &= n \oplus b = a = a^* = (n \oplus b)^*
\end{align*}
\]

So, \( \mathcal{L} \models Q_5 \). However, \( a \oplus 0 \neq 0 \oplus a \), so \( \mathcal{L} \not\models \forall x \forall y (x + y) = (y + x) \).

**Problem mar.2.** Expand \( \mathcal{L} \) of Example mar.2 to include \( \otimes \) and \( \ominus \) that interpret \( \times \) and \( < \). Show that your structure satisfies the remaining axioms of \( Q \),

\[
\begin{align*}
\forall x (x \times 0) &= 0 & (Q_6) \\
\forall x \forall y (x \times y') &= ((x \times y) + x) & (Q_7) \\
\forall x \forall y (x < y \leftrightarrow \exists z (z' + x) = y) & & (Q_8)
\end{align*}
\]

**Problem mar.3.** In \( \mathcal{L} \) of Example mar.2, \( a^* = a \) and \( b^* = b \). Is there a model of \( Q \) in which \( a^* = b \) and \( b^* = a \)?

We’ve explicitly constructed models of \( Q \) in which the non-standard elements live “beyond” the standard elements. In fact, that much is required by the axioms. A non-standard element \( x \) cannot be \( 0 \), since \( Q \vdash \forall x \neg x < 0 \) (see ??). Also, for every \( n \), \( Q \vdash \forall x (x < n' \rightarrow (x = 0 \lor x = 1 \lor \cdots \lor x = n)) \) (??), so we can’t have \( a \ominus n \) for any \( n > 0 \).

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**Bibliography**