mar.1 Models of PA

Any non-standard model of $\text{TA}$ is also one of $\text{PA}$. We know that non-standard models of $\text{TA}$ and hence of $\text{PA}$ exist. We also know that such non-standard models contain non-standard “numbers,” i.e., elements of the domain that are “beyond” all the standard “numbers.” But how are they arranged? How many are there? We’ve seen that models of the weaker theory $Q$ can contain as few as a single non-standard number. But these simple structures are not models of $\text{PA}$ or $\text{TA}$.

The key to understanding the structure of models of $\text{PA}$ or $\text{TA}$ is to see what facts are derivable in these theories. For instance, already $\text{PA}$ proves
\[ \forall x \neq x' \quad \text{and} \quad \forall x \forall y \ (x + y) = (y + x) , \]
so this rules out simple structures (in which these sentences are false) as models of $\text{PA}$.

Suppose $\mathcal{M}$ is a model of $\text{PA}$. Then if $\text{PA} \vdash \varphi$, $\mathcal{M} \models \varphi$. Let’s again use $z$ for $o^\mathcal{M}$, $*$ for $\rho^\mathcal{M}$, $\oplus$ for $+^\mathcal{M}$, $\otimes$ for $\times^\mathcal{M}$, and $\langle$ for $<^\mathcal{M}$. Any sentence $\varphi$ then states some condition about $z$, $*$, $\oplus$, $\otimes$, and $\langle$, and if $\mathcal{M} \models \varphi$ that condition must be satisfied. For instance, if $\mathcal{M} \models Q_1$, i.e., $\mathcal{M} \models \forall x \forall y \ (x' = y' \rightarrow x = y)$, then $*$ must be injective.

**Proposition mar.1.** In $\mathcal{M}$, $\otimes$ is a linear strict order, i.e., it satisfies:

1. $\not \exists x \otimes x$ for any $x \in |\mathcal{M}|$.
2. If $x \otimes y$ and $y \otimes z$ then $x \otimes z$.
3. For any $x \neq y$, $x \otimes y$ or $y \otimes x$

**Proof.** $\text{PA}$ proves:

1. $\forall x \neg x < x$
2. $\forall x \forall y \forall z \ ((x < y \land y < z) \rightarrow x < z)$
3. $\forall x \forall y \ ((x < y \lor y < x) \lor x = y)$

**Proposition mar.2.** $z$ is the least element of $|\mathcal{M}|$ in the $\otimes$-ordering. For any $x$, $x \otimes x^*$, and $x^*$ is the $\otimes$-least element with that property. For any $x$, there is a unique $y$ such that $y^* = x$. (We call $y$ the “predecessor” of $x$ in $\mathcal{M}$, and denote it by “$*$.”)

**Proof.** Exercise.

**Problem mar.1.** Find sentences in $\mathcal{L}_A$ derivable in $\text{PA}$ (and hence true in $\mathcal{M}$) which guarantee the properties of $z$, $*$, and $\otimes$ in Proposition mar.2

**Proposition mar.3.** All standard elements of $\mathcal{M}$ are less than (according to $\otimes$) all non-standard elements.
Proof. We’ll use $n$ as short for $Val^n(\bar{n})$, a standard element of $\mathcal{M}$. Already $Q$ proves that, for any $n \in \mathbb{N}$, $\forall x (x < \bar{n} \rightarrow (x = \overline{0} \lor x = \overline{1} \lor \cdots \lor x = \bar{n}))$. There are no elements that are $\emptyset z$. So if $n$ is standard and $x$ is non-standard, we cannot have $x \oplus n$. By definition, a non-standard element is one that isn’t $Val^n(\bar{n})$ for any $n \in \mathbb{N}$, so $x \neq n$ as well. Since $\emptyset$ is a linear order, we must have $n \oplus x$.

**Proposition mar.4.** Every nonstandard element $x$ of $|\mathcal{M}|$ is an element of the subset

$$\cdots \emptyset \emptyset \emptyset x \oplus \emptyset x \oplus x \oplus x \oplus x \oplus \cdots$$

We call this subset the block of $x$ and write it as $[x]$. It has no least and no greatest element. It can be characterized as the set of those $y \in |\mathcal{M}|$ such that, for some standard $n$, $x \oplus n = y$ or $y \oplus n = x$.

**Proof.** Clearly, such a set $[x]$ always exists since every element $y$ of $|\mathcal{M}|$ has a unique successor $y^*$ and unique predecessor "$y". For successive elements $y$, $y^*$ we have $y \odot y^*$ and $y^*$ is the $\ominus$-least element of $|\mathcal{M}|$ such that $y$ is $\ominus$-less than it. Since always "$y \odot y$ and $y \odot y^*$, $[x]$ has no least or greatest element. If $y \in [x]$ then $x \in [y]$, for then either $y^* \cdots x = x$ or $x \cdots y^* = y$. If $y^* \cdots x = x$ (with $n^*$’s), then $y \oplus n = x$ and conversely, since $PA \vdash \forall x x \cdots y = (x + \bar{n})$ (if $n$ is the number of $\forall$’s).

**Proposition mar.5.** If $[x] \neq [y]$ and $x \odot y$, then for any $u \in [x]$ and any $v \in [y]$, $u \odot v$.

**Proof.** Note that $PA \vdash \forall x \forall y (x < y \rightarrow (x' < y \lor x' = y))$. Thus, if $u \odot v$, we also have $u \odot n^* \odot v$ for any $n$ if $[u] \neq [v]$.

Any $u \in [x]$ is $\ominus y$: $x \odot y$ by assumption. If $u \odot x$, $u \odot y$ by transitivity. And if $x \odot u$ but $u \in [x]$, we have $u = x \odot n^*$ for some $n$, and so $u \odot y$ by the fact just proved.

Now suppose that $v \in [y]$ is $\ominus y$, i.e., $v \odot m^* = y$ for some standard $m$. This rules out $v \odot x$, otherwise $y = v \odot m^* \odot x$. Clearly also, $x \neq v$, otherwise $x \odot m^* = v \odot m^* = y$ and we would have $[x] = [y]$. So, $x \odot v$. But then also $x \odot n^* \odot v$ for any $n$. Hence, if $x \odot u$ and $u \in [x]$, we have $u \odot v$. If $u \odot x$ then $u \odot v$ by transitivity.

Lastly, if $y \odot v$, $u \odot v$ since, as we’ve shown, $u \odot y$ and $y \odot v$.

**Corollary mar.6.** If $[x] \neq [y]$, $[x] \cap [y] = \emptyset$.

**Proof.** Suppose $z \in [x]$ and $x \odot y$. Then $z \odot u$ for all $u \in [y]$. If $z \in [y]$, we would have $z \odot z$. Similarly if $y \odot x$.

**explanation**

This means that the blocks themselves can be ordered in a way that respects $\ominus$: $[x] \ominus [y]$ iff $x \ominus y$, or, equivalently, if $u \odot v$ for any $u \in [x]$ and $v \in [y]$. Clearly, the standard block $[\emptyset]$ is the least block. It intersects with no non-standard block, and no two non-standard blocks intersect either. Specifically, you cannot “reach” a different block by taking repeated successors or predecessors.
Proposition mar.7. If \( x \) and \( y \) are non-standard, then \( x \oplus x \oplus y \) and \( x \oplus y \notin [x] \).

Proof. If \( y \) is nonstandard, then \( y \neq z \). \( \mathsf{PA} \vdash \forall x (y \neq 0 \rightarrow x < (x + y)) \). Now suppose \( x \oplus y \in [x] \). Since \( x \oplus x \oplus y \), we would have \( x \oplus n^* = x \oplus y \). But \( \mathsf{PA} \vdash \forall x \forall y \forall z ((x + y) = (x + z) \rightarrow y = z) \) (the cancellation law for addition). This would mean \( y = n^* \) for some standard \( n \); but \( y \) is assumed to be non-standard. \( \square \)

Proposition mar.8. There is no least non-standard block.

Proof. \( \mathsf{PA} \vdash \forall x \exists y ((y + y) = x \lor (y + y)' = x) \), i.e., that every \( x \) is divisible by 2 (possibly with remainder 1). If \( x \) is non-standard, so is \( y \). By the preceding proposition, \( y \oplus y \oplus y \) and \( y \oplus y \notin [y] \). Then also \( y \oplus (y \oplus y)^* \) and \( (y \oplus y)^* \notin [y] \). But \( x = y \oplus y \) or \( x = (y \oplus y)^* \), so \( y \oplus x \) and \( y \notin [x] \). \( \square \)

Proposition mar.9. There is no largest block.

Proof. Exercise. \( \square \)

Problem mar.2. Show that in a non-standard model of \( \mathsf{PA} \), there is no largest block.

Proposition mar.10. The ordering of the blocks is dense. That is, if \( x \oplus y \) and \( [x] \neq [y] \), then there is a block \([z]\) distinct from both that is between them.

Proof. Suppose \( x \oplus y \). As before, \( x \oplus y \) is divisible by two (possibly with remainder): there is a \( z \in [M] \) such that either \( x \oplus y = z \oplus z \) or \( x \oplus y = (z \oplus z)^* \). The element \( z \) is the “average” of \( x \) and \( y \), and \( x \oplus z \) and \( z \oplus y \). \( \square \)

Problem mar.3. Write out a detailed proof of Proposition mar.10. Which sentence must \( \mathsf{PA} \) derive in order to guarantee the existence of \( z \)? Why is \( x \oplus z \) and \( z \oplus y \), and why is \( [x] \neq [z] \) and \( [z] \neq [y] \)?

The non-standard blocks are therefore ordered like the rationals: they form a denumerable dense linear ordering without endpoints. One can show that any two such denumerable orderings are isomorphic. It follows that for any two enumerable non-standard models \( M_1 \) and \( M_2 \) of true arithmetic, their reducts to the language containing \(< \) and \( = \) only are isomorphic. Indeed, an isomorphism \( h \) can be defined as follows: the standard parts of \( M_1 \) and \( M_2 \) are isomorphic to the standard model \( \mathbb{N} \) and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore isomorphic; an isomorphism of the blocks can be extended to an isomorphism within the blocks by matching up arbitrary elements in each, and then taking the image of the successor of \( x \) in \( M_1 \) to be the successor of the image of \( x \) in \( M_2 \). Note that it does not follow that \( M_1 \) and \( M_2 \) are isomorphic in the full language of arithmetic (indeed, isomorphism is always
relative to a language), as there are non-isomorphic ways to define addition and multiplication over $|\mathcal{M}_1|$ and $|\mathcal{M}_2|$. (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be 2.)

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Bibliography