

mar.1 Models of PA

Any non-standard model of **TA** is also one of **PA**. We know that non-standard models of **TA** and hence of **PA** exist. We also know that such non-standard models contain non-standard “numbers,” i.e., **elements** of the domain that are “beyond” all the standard “numbers.” But how are they arranged? How many are there? We’ve seen that models of the weaker theory **Q** can contain as few as a single non-standard number. But these simple **structures** are not models of **PA** or **TA**. explanation

The key to understanding the structure of models of **PA** or **TA** is to see what facts are **derivable** in these theories. For instance, already **PA** proves that $\forall x x \neq x'$ and $\forall x \forall y (x + y) = (y + x)$, so this rules out simple structures (in which these **sentences** are false) as models of **PA**.

Suppose \mathfrak{M} is a model of **PA**. Then if $\mathbf{PA} \vdash \varphi$, $\mathfrak{M} \models \varphi$. Let’s again use \mathbf{z} for $0^{\mathfrak{M}}$, $*$ for $1^{\mathfrak{M}}$, \oplus for $+$, \otimes for \times , and \ominus for $<$. Any **sentence** φ then states some condition about \mathbf{z} , $*$, \oplus , \otimes , and \ominus , and if $\mathfrak{M} \models \varphi$ that condition must be satisfied. For instance, if $\mathfrak{M} \models Q_1$, i.e., $\mathfrak{M} \models \forall x \forall y (x' = y' \rightarrow x = y)$, then $*$ must be **injective**.

Proposition mar.1. *In \mathfrak{M} , \ominus is a linear strict order, i.e., it satisfies:*

1. Not $x \ominus x$ for any $x \in |\mathfrak{M}|$.
2. If $x \ominus y$ and $y \ominus z$ then $x \ominus z$.
3. For any $x \neq y$, $x \ominus y$ or $y \ominus x$

Proof. **PA** proves:

1. $\forall x \neg x < x$
2. $\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$
3. $\forall x \forall y ((x < y \vee y < x) \vee x = y)$ □

*mod:mar:mpa:
prop:M-discrete*

Proposition mar.2. *\mathbf{z} is the least **element** of $|\mathfrak{M}|$ in the \ominus -ordering. For any x , $x \ominus x^*$, and x^* is the \ominus -least **element** with that property. For any x , there is a unique y such that $y^* = x$. (We call y the “predecessor” of x in \mathfrak{M} , and denote it by $*x$.)*

Proof. Exercise. □

Problem mar.1. Find **sentences** in \mathcal{L}_A **derivable** in **PA** (and hence true in \mathfrak{N}) which guarantee the properties of \mathbf{z} , $*$, and \ominus in **Proposition mar.2**

Proposition mar.3. *All standard **elements** of \mathfrak{M} are less than (according to \ominus) all non-standard **elements**.*

Proof. We'll use n as short for $\text{Val}^{\mathfrak{M}}(\bar{n})$, a standard **element** of \mathfrak{M} . Already **Q** proves that, for any $n \in \mathbb{N}$, $\forall x (x < \bar{n}' \rightarrow (x = \bar{0} \vee x = \bar{1} \vee \dots \vee x = \bar{n}))$. There are no **elements** that are $\ominus \mathbf{z}$. So if n is standard and x is non-standard, we cannot have $x \ominus n$. By definition, a non-standard element is one that isn't $\text{Val}^{\mathfrak{M}}(\bar{n})$ for any $n \in \mathbb{N}$, so $x \neq n$ as well. Since \ominus is a linear order, we must have $n \ominus x$. \square

Proposition mar.4. *Every nonstandard **element** x of $|\mathfrak{M}|$ is an element of the subset*

$$\dots^{***} x \ominus^{**} x \ominus^* x \ominus x \ominus x \ominus x^* \ominus x^{**} \ominus x^{***} \ominus \dots$$

*We call this subset the block of x and write it as $[x]$. It has no least and no greatest **element**. It can be characterized as the set of those $y \in |\mathfrak{M}|$ such that, for some standard n , $x \oplus n = y$ or $y \oplus n = x$.*

Proof. Clearly, such a set $[x]$ always exists since every **element** y of $|\mathfrak{M}|$ has a unique successor y^* and unique predecessor *y . For successive **elements** y , y^* we have $y \ominus y^*$ and y^* is the \ominus -least **element** of $|\mathfrak{M}|$ such that y is \ominus -less than it. Since always $^*y \ominus y$ and $y \ominus y^*$, $[x]$ has no least or greatest **element**. If $y \in [x]$ then $x \in [y]$, for then either $y^{***} = x$ or $x^{***} = y$. If $y^{***} = x$ (with n $*$'s), then $y \oplus n = x$ and conversely, since $\mathbf{PA} \vdash \forall x x' \dots' = (x + \bar{n})$ (if n is the number of $'$'s). \square

Proposition mar.5. *If $[x] \neq [y]$ and $x \ominus y$, then for any $u \in [x]$ and any $v \in [y]$, $u \ominus v$.*

Proof. Note that $\mathbf{PA} \vdash \forall x \forall y (x < y \rightarrow (x' < y \vee x' = y))$. Thus, if $u \ominus v$, we also have $u \oplus n^* \ominus v$ for any n if $[u] \neq [v]$.

Any $u \in [x]$ is $\ominus y$: $x \ominus y$ by assumption. If $u \ominus x$, $u \ominus y$ by transitivity. And if $x \ominus u$ but $u \in [x]$, we have $u = x \oplus n^*$ for some n , and so $u \ominus y$ by the fact just proved.

Now suppose that $v \in [y]$ is $\ominus y$, i.e., $v \oplus m^* = y$ for some standard m . This rules out $v \ominus x$, otherwise $y = v \oplus m^* \ominus x$. Clearly also, $x \neq v$, otherwise $x \oplus m^* = v \oplus m^* = y$ and we would have $[x] = [y]$. So, $x \ominus v$. But then also $x \oplus n^* \ominus v$ for any n . Hence, if $x \ominus u$ and $u \in [x]$, we have $u \ominus v$. If $u \ominus x$ then $u \ominus v$ by transitivity.

Lastly, if $y \ominus v$, $u \ominus v$ since, as we've shown, $u \ominus y$ and $y \ominus v$. \square

Corollary mar.6. *If $[x] \neq [y]$, $[x] \cap [y] = \emptyset$.*

Proof. Suppose $z \in [x]$ and $x \ominus y$. Then $z \ominus u$ for all $u \in [y]$. If $z \in [y]$, we would have $z \ominus z$. Similarly if $y \ominus x$. \square

explanation

This means that the blocks themselves can be ordered in a way that respects \ominus : $[x] \ominus [y]$ iff $x \ominus y$, or, equivalently, if $u \ominus v$ for any $u \in [x]$ and $v \in [y]$. Clearly, the standard block $[0]$ is the least block. It intersects with no non-standard block, and no two non-standard blocks intersect either. Specifically, you cannot "reach" a different block by taking repeated successors or predecessors.

Proposition mar.7. *If x and y are non-standard, then $x \otimes x \oplus y$ and $x \oplus y \notin [x]$.*

Proof. If y is nonstandard, then $y \neq \mathbf{z}$. $\mathbf{PA} \vdash \forall x (y \neq 0 \rightarrow x < (x + y))$. Now suppose $x \oplus y \in [x]$. Since $x \otimes x \oplus y$, we would have $x \oplus n^* = x \oplus y$. But $\mathbf{PA} \vdash \forall x \forall y \forall z ((x + y) = (x + z) \rightarrow y = z)$ (the cancellation law for addition). This would mean $y = n^*$ for some standard n ; but y is assumed to be non-standard. \square

Proposition mar.8. *There is no least non-standard block.*

Proof. $\mathbf{PA} \vdash \forall x \exists y ((y + y) = x \vee (y + y)' = x)$, i.e., that every x is divisible by 2 (possibly with remainder 1). If x is non-standard, so is y . By the preceding proposition, $y \otimes y \oplus y$ and $y \oplus y \notin [y]$. Then also $y \otimes (y \oplus y)^*$ and $(y \oplus y)^* \notin [y]$. But $x = y \oplus y$ or $x = (y \oplus y)^*$, so $y \otimes x$ and $y \notin [x]$. \square

Proposition mar.9. *There is no largest block.*

Proof. Exercise. \square

Problem mar.2. Show that in a non-standard model of \mathbf{PA} , there is no largest block.

*mod:mar:mpa:
prop:blocks-dense*

Proposition mar.10. *The ordering of the blocks is dense. That is, if $x \otimes y$ and $[x] \neq [y]$, then there is a block $[z]$ distinct from both that is between them.*

Proof. Suppose $x \otimes y$. As before, $x \oplus y$ is divisible by two (possibly with remainder): there is a $z \in |\mathfrak{M}|$ such that either $x \oplus y = z \oplus z$ or $x \oplus y = (z \oplus z)^*$. The element z is the “average” of x and y , and $x \otimes z$ and $z \otimes y$. \square

Problem mar.3. Write out a detailed proof of **Proposition mar.10**. Which sentence must \mathbf{PA} derive in order to guarantee the existence of z ? Why is $x \otimes z$ and $z \otimes y$, and why is $[x] \neq [z]$ and $[z] \neq [y]$?

The non-standard blocks are therefore ordered like the rationals: they form a **denumerable** dense linear ordering without endpoints. One can show that any two such **denumerable** orderings are isomorphic. It follows that for any two **enumerable** non-standard models \mathfrak{M}_1 and \mathfrak{M}_2 of true arithmetic, their reducts to the language containing $<$ and $=$ only are isomorphic. Indeed, an isomorphism h can be defined as follows: the standard parts of \mathfrak{M}_1 and \mathfrak{M}_2 are isomorphic to the standard model \mathfrak{N} and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore isomorphic; an isomorphism of the blocks can be extended to an isomorphism *within* the blocks by matching up arbitrary elements in each, and then taking the image of the successor of x in \mathfrak{M}_1 to be the successor of the image of x in \mathfrak{M}_2 . Note that it does *not* follow that \mathfrak{M}_1 and \mathfrak{M}_2 are isomorphic in the full language of arithmetic (indeed, isomorphism is always explanation

relative to a language), as there are non-isomorphic ways to define addition and multiplication over \mathfrak{M}_1 and \mathfrak{M}_2 . (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be 2.)

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Bibliography