Chapter udf

Models of Arithmetic

mar.1 Introduction

The *standard model* of aritmetic is the structure $\mathfrak{N}$ with $|\mathfrak{N}| = \mathbb{N}$ in which o, $\prime$, $+$, $\times$, and $<$ are interpreted as you would expect. That is, o is 0, $\prime$ is the successor function, $+$ is interpeted as addition and $\times$ as multiplication of the numbers in $\mathbb{N}$. Specifically,

$$\begin{aligned}
\mathsf{o}^{\mathfrak{N}} &= 0 \\
\prime^{\mathfrak{N}}(n) &= n + 1 \\
+^{\mathfrak{N}}(n, m) &= n + m \\
\times^{\mathfrak{N}}(n, m) &= nm
\end{aligned}$$

Of course, there are structures for $\mathcal{L}_A$ that have domains other than $\mathbb{N}$. For instance, we can take $\mathfrak{M}$ with domain $|\mathfrak{M}| = \{a\}^*$ (the finite sequences of the single symbol a, i.e., $\emptyset$, a, aa, aaa, ...), and interpretations

$$\begin{aligned}
\mathsf{o}^{\mathfrak{M}} &= \emptyset \\
\prime^{\mathfrak{M}}(s) &= s \frown a \\
+^{\mathfrak{M}}(n, m) &= a^{n+m} \\
\times^{\mathfrak{M}}(n, m) &= a^{nm}
\end{aligned}$$

These two structures are "essentially the same" in the sense that the only difference is the elements of the domains but not how the elements of the domains are related among each other by the interpretation functions. We say that the two structures are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in $\mathfrak{N}$ also has models that are not isomorphic to $\mathfrak{N}$. Such structures are called *non-standard*. The interesting thing about them is that while the elements of a standard model (i.e., $\mathfrak{N}$, but also all structures isomorphic to it) are exhausted by the values of the standard numerals $\overline{n}$, i.e.,

$$|\mathfrak{N}| = \{\mathrm{Val}^{\mathfrak{N}}(\overline{n}) : n \in \mathbb{N}\}$$

that isn't the case in non-standard models: if $\mathfrak{M}$ is non-standard, then there is at least one $x \in |\mathfrak{M}|$ such that $x \neq \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all n.

These non-standard elements are pretty neat: they are "infinite natural numbers." But their existence also explains, in a sense, the incompleteness phenomena. Cconsider an example, e.g., the consistency statement for Peano arithmetic, $\mathsf{Con}_{\mathbf{PA}}$, i.e., $\lnot \exists x \, \mathsf{Prf}_{\mathbf{PA}}(x, \ulcorner \bot \urcorner)$. Since $\mathbf{PA}$ neither proves $\mathsf{Con}_{\mathbf{PA}}$ nor $\lnot \mathsf{Con}_{\mathbf{PA}}$, either can be consistently added to $\mathbf{PA}$. Since $\mathbf{PA}$ is consistent, $\mathfrak{N} \vDash \mathsf{Con}_{\mathbf{PA}}$, and consequently $\mathfrak{N} \nvDash \lnot \mathsf{Con}_{\mathbf{PA}}$. So $\mathfrak{N}$ is *not* a model of $\mathbf{PA} \cup \{\lnot \mathsf{Con}_{\mathbf{PA}}\}$, and all its models must be nonstandard. Models of $\mathbf{PA} \cup \{\lnot \mathsf{Con}_{\mathbf{PA}}\}$ must contain some element that serves as the witness that makes $\exists x \, \mathsf{Prf}_{\mathbf{PA}}(\ulcorner \bot \urcorner)$ true, i.e., a Gödel number of a derivation of a contradiction from $\mathbf{PA}$. Such an element can't be standard—since $\mathbf{PA} \vdash \lnot \mathsf{Prf}_{\mathbf{PA}}(\overline{n}, \ulcorner \bot \urcorner)$ for every n.

mar.2 Standard Models of Arithmetic

The language of arithmetic $\mathcal{L}_A$ is obviously intended to be about numbers, specifically, about natural numbers. So, "the" standard model $\mathfrak{N}$ is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two structures taht are isomorphic share those. So we can be a bit more liberal, and consider any structure that is isomorphic to $\mathfrak{N}$ "standard." mod:mar:stm:sec

Definition mar.1. A structure for $\mathcal{L}_A$ is *standard* if it is isomorphic to $\mathfrak{N}$.

Proposition mar.2. *If a structure $\mathfrak{M}$ standard, its domain is the set of values of the standard numerals, i.e.,* mod:mar:stm:prop:standard-domain

$$|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$$

Proof. Clearly, every $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \in |\mathfrak{M}|$. We just have to show that every $x \in |\mathfrak{M}|$ is equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some n. Since $\mathfrak{M}$ is standard, it is isomorphic to $\mathfrak{N}$. Suppose $g \colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism. Then $g(n) = g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. But for every $x \in |\mathfrak{M}|$, there is an $n \in \mathbb{N}$ such that $g(n) = x$, since g is surjective. □

explanation If a structure $\mathfrak{M}$ for $\mathcal{L}_A$ is standard, the elements of its domain can all be named by the standard numerals $\overline{0}$, $\overline{1}$, $\overline{2}$, ..., i.e., the terms o, o', o'', etc. Of course, this does not mean that the elements of $|\mathfrak{M}|$ *are* the numbers, just that we can pick them out the same way we can pick out the numbers in $|\mathfrak{N}|$.

Problem mar.1. Show that the converse of Proposition mar.2 is false, i.e., give an example of a structure $\mathfrak{M}$ with $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$ that is not isomorphic to $\mathfrak{N}$.

Proposition mar.3. *If $\mathfrak{M} \vDash \mathbf{Q}$, and $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$, then $\mathfrak{M}$ is standard.* mod:mar:stm:prop:thq-standard

Proof. We have to show that $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$. Consider the function $g\colon \mathbb{N} \to |\mathfrak{M}|$ defined by $g(n) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. By the hypothesis, g is surjective. It is also injective: $\mathbf{Q} \vdash \overline{n} \neq \overline{m}$ whenever $n \neq m$. Thus, since $\mathfrak{M} \vDash \mathbf{Q}$, $\mathfrak{M} \vDash \overline{n} \neq \overline{m}$, whenever $n \neq m$. Thus, if $n \neq m$, then $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \neq \mathrm{Val}^{\mathfrak{M}}(\overline{m})$, i.e., $g(n) \neq g(m)$.

We also have to verify that g is an isomorphism.

1. We have $g(\mathsf{o}^{\mathfrak{N}}) = g(0)$ since, $\mathsf{o}^{\mathfrak{N}} = 0$. By definition of g, $g(0) = \mathrm{Val}^{\mathfrak{M}}(\overline{0})$. But $\overline{0}$ is just o, and the value of a term which happens to be a constant symbol is given by what the structure assigns to that constant symbol, i.e., $\mathrm{Val}^{\mathfrak{M}}(\mathsf{o}) = \mathsf{o}^{\mathfrak{M}}$. So we have $g(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$ as required.

2. $g({\prime}^{\mathfrak{N}}(n)) = g(n+1)$, since $\prime$ in $\mathfrak{N}$ is the successor function on $\mathbb{N}$. Then, $g(n+1) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+1})$ by definition of g. But $\overline{n+1}$ is the same term as $\overline{n}'$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n}')$. By the definition of the value function, this is $= {\prime}^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ we get $g({\prime}^{\mathfrak{N}}(n)) = {\prime}^{\mathfrak{M}}(g(n))$.

3. $g(+^{\mathfrak{N}}(n,m)) = g(n+m)$, since $+$ in $\mathfrak{N}$ is the addition function on $\mathbb{N}$. Then, $g(n+m) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+m})$ by definition of g. But $\mathbf{Q} \vdash \overline{n+m} = (\overline{n} + \overline{m})$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+m}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n} + \overline{m})$. By the definition of the value function, this is $= +^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ and $\mathrm{Val}^{\mathfrak{M}}(\overline{m}) = g(m)$, we get $g(+^{\mathfrak{N}}(n,m)) = +^{\mathfrak{M}}(g(n), g(m))$.

4. $g(\times^{\mathfrak{N}}(n,m)) = \times^{\mathfrak{M}}(g(n), g(m))$: Exercise.

5. $\langle n, m\rangle \in <^{\mathfrak{N}}$ iff $n < m$. If $n < m$, then $\mathbf{Q} \vdash \overline{n} < \overline{m}$, and also $\mathfrak{M} \vDash \overline{n} < \overline{m}$. Thus $\langle \mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m})\rangle \in <^{\mathfrak{M}}$, i.e., $\langle g(n), g(m)\rangle \in <^{\mathfrak{M}}$. If $n \not< m$, then $\mathbf{Q} \vdash \neg \overline{n} < \overline{m}$, and consequently $\mathfrak{M} \nvDash \overline{n} < \overline{m}$. Thus, as before, $\langle g(n), g(m)\rangle \notin <^{\mathfrak{M}}$. Together, we get: $\langle n, m\rangle \in <^{\mathfrak{N}}$ iff $\langle g(n), g(m)\rangle \in <^{\mathfrak{M}}$.

□

The function g is the most obvious way of defining a mapping from $\mathbb{N}$ to the domain of any other structure $\mathfrak{M}$ for $\mathcal{L}_A$, since every such $\mathfrak{M}$ contains elements named by $\overline{0}$, $\overline{1}$, $\overline{2}$, etc. So it isn't surprising that if $\mathfrak{M}$ makes at least some basic statements about the $\overline{n}$'s true in the same way that $\mathfrak{N}$ does, and g is also bijective, then g will turn into an isomorphism. In fact, if $|\mathfrak{M}|$ contains no elements other than what the $\overline{n}$'s name, it's the only one. explanation

mod:mar:stm:prop:thq-unique-iso **Proposition mar.4.** *If $\mathfrak{M}$ is standard, then g from the proof of Proposition mar.3 is the only isomorphism from $\mathfrak{N}$ to $\mathfrak{M}$.*

Proof. Suppose $h\colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism between $\mathfrak{N}$ and $\mathfrak{M}$. We show that $g = h$ by induction on n. If $n = 0$, then $g(0) = \mathsf{o}^{\mathfrak{M}}$ by definition of g. But since h is an isomorphism, $h(0) = h(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$, so $g(0) = h(0)$.

Now consider the case for $n+1$. We have

$$\begin{aligned}
g(n+1) &= \mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) \text{ by definition of } g\\
&= \mathrm{Val}^{\mathfrak{M}}(\overline{n}')\\
&= \prime^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}))\\
&= \prime^{\mathfrak{M}}(g(n)) \text{ by definition of } g\\
&= \prime^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis}\\
&= h(\prime^{\mathfrak{N}}(n)) \text{ since } h \text{ is an isomorphism}\\
&= h(n+1)
\end{aligned}$$

□

explanation For any denumerable set X, there's a bijection between $\mathbb{N}$ and X, so every such set X is potentially the domain of a standard model. In fact, once you pick an object $z \in X$ and a suitable function $s\colon X \to X$ as $\mathsf{o}^{\mathfrak{X}}$ and $\prime^{\mathfrak{X}}$, the interpretation of $+$, $\times$, and $<$ is already fixed. Only functions $s = \prime^{\mathfrak{X}}$ that are both injective and surjective are suitable in a standard model. It has to be injective since the successor function in $\mathfrak{N}$ is, and that $\prime$ is injective is expressed by a sentence true in $\mathfrak{N}$ which $\mathfrak{X}$ thus also has to make true. It has to be surjective because otherwise there would be some $x \in X$ not in the domain of s, i.e., the sentence $\forall x\, \exists y\, y' = x$ would be false—but it is true in $\mathfrak{N}$.

mar.3 Non-Standard Models

explanation We call a structure for $\mathcal{L}_A$ standard if it is isomorphic to $\mathfrak{N}$. If a structure isn't isomorphic to $\mathfrak{N}$, it is called non-standard.

Definition mar.5. A structure $\mathfrak{M}$ for $\mathcal{L}_A$ is *non-standard* if it is not isomorphic to $\mathfrak{N}$. The elements $x \in |\mathfrak{M}|$ which are equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some $n \in \mathbb{N}$ are called *standard numbers* (of $\mathfrak{M}$), and those not, *non-standard numbers*.

explanation By Proposition mar.2, any standard structure for $\mathcal{L}_A$ contains only standard elements. Consequently, a non-standard structure must contain at least one non-standard element. In fact, the existence of a non-standard element guarantees that the structure is non-standard.

Proposition mar.6. *If a structure $\mathfrak{M}$ for $\mathcal{L}_A$ contains a non-standard number, $\mathfrak{M}$ is non-standard.*

Proof. Suppose not, i.e., suppose $\mathfrak{M}$ standard but contains a non-standard number x. Let $g\colon \mathbb{N} \to |\mathfrak{M}|$ be an isomorphism. It is easy to see (by induction on n) that $g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. In other words, g maps standard numbers of $\mathfrak{N}$ to standard numbers of $\mathfrak{M}$. If $\mathfrak{M}$ contains a non-standard number, g cannot be surjective, contrary to hypothesis. □

Problem mar.2. Recall that $\mathbf{Q}$ contains the axioms

$$\forall x\,\forall y\,(x' = y' \rightarrow x = y) \tag{Q_1}$$

$$\forall x\,\mathsf{o} \neq x' \tag{Q_2}$$

$$\forall x\,(x \neq \mathsf{o} \rightarrow \exists y\, x = y') \tag{Q_3}$$

Give structures $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ such that

1. $\mathfrak{M}_1 \vDash Q_1$, $\mathfrak{M}_1 \vDash Q_2$, $\mathfrak{M}_1 \nvDash Q_3$;
2. $\mathfrak{M}_2 \vDash Q_1$, $\mathfrak{M}_2 \nvDash Q_2$, $\mathfrak{M}_2 \vDash Q_3$; and
3. $\mathfrak{M}_3 \nvDash Q_1$, $\mathfrak{M}_3 \vDash Q_2$, $\mathfrak{M}_3 \vDash Q_3$;

Obviously, you just have to specify $\mathsf{o}^{\mathfrak{M}_i}$ and $\prime^{\mathfrak{M}_i}$ for each.

It is easy enough to specify non-standard structures for $\mathcal{L}_A$. For instance, take the structure with domain $\mathbb{Z}$ and interpret all non-logical symbols as usual. Since negative numbers are not values of $\overline{n}$ for any n, this structure is non-standard. Of course, it will not be a *model* of arithmetic in the sense that it makes the same sentences true as $\mathfrak{N}$. For instance, $\forall x\, x' \neq \mathsf{o}$ is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem. explanation

Proposition mar.7. *Let* $\mathbf{TA} = \{\varphi : \mathfrak{N} \vDash \varphi\}$ *be the theory of* $\mathbf{N}$. $\mathbf{TA}$ *has an enumerable non-standard model.*

Proof. Expand $\mathcal{L}_A$ by a new constant symbol c and consider the set of sentences

$$\Gamma = \mathbf{TA} \cup \{c \neq \overline{0}, c \neq \overline{1}, c \neq \overline{2}, \dots\}$$

Any model $\mathfrak{M}^c$ of Γ would contain an element $x = c^{\mathfrak{M}}$ which is non-standard, since $x \neq \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all $n \in \mathbb{N}$. Also, obviously, $\mathfrak{M}^c \vDash \mathbf{TA}$, since $\mathbf{TA} \subseteq \Gamma$. If we turn $\mathfrak{M}^c$ into a structure $\mathfrak{M}$ for $\mathcal{L}_A$ simply by forgetting about c, its domain still contains the non-standard x, and also $\mathfrak{M} \vDash \mathbf{TA}$. The latter is guaranteed since c does not occur in $\mathbf{TA}$. So, it suffices to show that Γ has a model.

We use the compactness theorem to show that Γ has a model. If every finite subset of Γ is satisfiable, so is Γ. Consider any finite subset $\Gamma_0 \subseteq \Gamma$. Γ_0 includes some sentences of $\mathbf{TA}$ and some of the form $c \neq \overline{n}$, but only finitely many. Suppose k is the largest number so that $c \neq \overline{k} \in \Gamma_0$. Define $\mathfrak{N}_k$ by expanding $\mathfrak{N}$ to include the interpretation $c^{\mathfrak{N}_k} = k + 1$. $\mathfrak{N}_k \vDash \Gamma_0$: if $\varphi \in \mathbf{TA}$, $\mathfrak{N}_k \vDash \varphi$ since $\mathfrak{N}_k$ is just like $\mathfrak{N}$ in all respects except c, and c does not occur in φ. And $\mathfrak{N}_k \vDash c \neq \overline{n}$, since $n \leq k$, and $\mathrm{Val}^{\mathfrak{N}_k}(c) = k + 1$. Thus, every finite subset of Γ is satisfiable. □

mar.4 Models of Q

explanation We know that there are non-standard structures that make the same sentences true as $\mathfrak{N}$ does, i.e., is a model of **TA**. Since $\mathfrak{N} \vDash \mathbf{Q}$, any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g., $\mathbf{Q} \nvDash \forall x \, \forall y \, (x + y) = (y + x)$. Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make $\forall x \, \forall y \, (x + y) = (y + x)$ false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly.

Example mar.8. Consider the structure $\mathfrak{K}$ with domain $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations mod:mar:mdq: ex:model-K-of-Q

$$\begin{aligned}
\mathsf{o}^{\mathfrak{K}} &= 0 \\
\prime^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\
+^{\mathfrak{K}}(x, y) &= \begin{cases} x+y & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
\times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
<^{\mathfrak{K}} &= \{\langle x, y\rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a\rangle : x \in |\mathfrak{K}|\}
\end{aligned}$$

To show that $\mathfrak{K} \vDash \mathbf{Q}$ we have to verify that all axioms of **Q** are true in $\mathfrak{K}$. For convenience, let's write x^* for $\prime^{\mathfrak{K}}(x)$ (the "successor" of x in $\mathfrak{K}$), $x \oplus y$ for $+^{\mathfrak{K}}(x, y)$ (the "sum" of x and y in $\mathfrak{K}$, $x \otimes y$ for $\times^{\mathfrak{K}}(x, y)$ (the "product" of x and y in $\mathfrak{K}$), and $x \lessdot y$ for $\langle x, y\rangle \in <^{\mathfrak{K}}$. With these abbreviations, we can give the operations in $\mathfrak{K}$ more perspicuously as

x	x^*
n	$n+1$
a	a

$x \oplus y$	m	a
n	$n+m$	a
a	a	a

$x \otimes y$	m	a
n	nm	a
a	a	a

We have $n \lessdot m$ iff $n < m$ for $n, m \in \mathbb{N}$ and $x \lessdot a$ for all $x \in |\mathfrak{K}|$.

$\mathfrak{K} \vDash \forall x \, \forall y \, (x' = y' \rightarrow x = y)$ since $*$ is injective. $\mathfrak{K} \vDash \forall x \, \mathsf{o} \neq x'$ since 0 is not a $*$-successor in $\mathfrak{K}$. $\mathfrak{N} \vDash \forall x \, (x \neq \mathsf{o} \rightarrow \exists y \, x = y')$ since for every $n > 0$, $n = (n-1)^*$, and $a = a^*$.

$\mathfrak{K} \vDash \forall x \, (x + \mathsf{o}) = x$ since $n \oplus 0 = n + 0 = n$, and $a \oplus 0 = a$ by definition of $\oplus$. $\mathfrak{K} \vDash \forall x \, \forall y \, (x + y') = (x + y)'$ is a bit trickier. If n, m are both standard, we have:

$$(n \oplus m^*) = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*$$

since $\oplus$ and * agree with $+$ and $\prime$ on standard numbers. Now suppose $x \in |\mathfrak{K}|$. Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if $y \in |\mathfrak{K}|$ but $x = a$. Here we also have to distinguish cases according to whether $y = n$ is standard or $y = b$:

$$(a \oplus n^*) = (a \oplus (n+1)) = a = a^* = (x \oplus n)^*$$
$$(a \oplus a^*) = (a \oplus a) = a = a^* = (x \oplus a)^*$$

This is of course a bit more detailed than needed. For instance, since $a \oplus z = a$ whatever z is, we can immediately conclude $a \oplus a^* = a$. The remaining axioms can be verified the same way.

$\mathfrak{K}$ is thus a model of $\mathbf{Q}$. Its "addition" $\oplus$ is also commutative. But there are other sentences true in $\mathfrak{N}$ but false in $\mathfrak{K}$, and vice versa. For instance, $a \otimes a$, so $\mathfrak{K} \vDash \exists x\, x < x$ and $\mathfrak{K} \nvDash \forall x\, \neg x < x$. This shows that $\mathbf{Q} \nvdash \forall x\, \neg x < x$.

Problem mar.3. Prove that $\mathfrak{K}$ from Example mar.8 satisifies the remaining axioms of $\mathbf{Q}$,

$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$
$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$
$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (x + z' = y)) \tag{Q_8}$$

Find a sentence only involving $'$ true in $\mathfrak{N}$ but false in $\mathfrak{K}$.

mod:mar:mdq: ex:model-L-of-Q **Example mar.9.** Consider the structure $\mathfrak{L}$ with domain $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$ and interpretations $'^{\mathfrak{L}} = *$, $+^{\mathfrak{L}} = \oplus$ given by

x	x^*
n	$n+1$
a	a
b	b

$x \oplus y$	m	a	b
n	$n+m$	b	a
a	a	b	a
b	b	b	a

Since $*$ is injective, 0 is not in its range, and every $x \in |\mathfrak{L}|$ other than 0 is, axioms Q_1–Q_3 are true in $\mathfrak{L}$. For any x, $x \oplus 0 = x$, so Q_4 is true as well. For Q_5, consider $x \oplus y^*$ and $(x \oplus y)^*$. They are equal if x and y are both standard, since then $*$ and $\oplus$ agree with $'$ and $+$. If x is non-standard, and y is standard, we have $x \oplus y^* = x = x^* = (x \oplus y)^*$. If x and y are both non-standard, we have four cases:

$$a \oplus a^* = b = b^* = (a \oplus a)^*$$
$$b \oplus b^* = a = a^* = (b \oplus b)^*$$
$$b \oplus a^* = b = b^* = (b \oplus y)^*$$
$$a \oplus b^* = a = a^* = (a \oplus b)^*$$

If x is standard, but y is non-standard, we have

$$n \oplus a^* = n \oplus a = b = b^* = (n \oplus a)^*$$
$$n \oplus b^* = n \oplus b = a = a^* = (n \oplus b)^*$$

So, $\mathfrak{L} \vDash Q_5$. However, $a \oplus 0 \neq 0 \oplus a$, so $\mathfrak{L} \nvDash \forall x\, \forall y\, (x + y) = (y + x)$.

Problem mar.4. Expand $\mathfrak{L}$ of Example mar.9 to include $\otimes$ and $\lessdot$ that interpret $\times$ and $<$. Show that your structure satisifies the remaining axioms of $\mathbf{Q}$,

$$\forall x\,(x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$
$$\forall x\,\forall y\,(x \times y') = ((x \times y) + x) \tag{Q_7}$$
$$\forall x\,\forall y\,(x < y \leftrightarrow \exists z\,(x + z' = y)) \tag{Q_8}$$

Problem mar.5. In $\mathfrak{L}$ of Example mar.9, $a^* = a$ and $b^* = b$. Is there a model of $\mathbf{Q}$ in which $a^* = b$ and $b^* = a$?

explanation

We've explicitly constructed models of $\mathbf{Q}$ in which the non-standard elements live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard element x cannot be $\lessdot 0$. Otherwise, for some z, $x \oplus z^* = 0$ by $Q8$. But then $0 = x \oplus z^* = (x \oplus z)^*$ by Q_5, contradicting Q_2. Also, for every n, $\mathbf{Q} \vdash \forall x\,(x < \overline{n}' \rightarrow (x = \overline{0} \lor x = \overline{1} \lor \dots \lor x = \overline{n}))$, so we can't have $a \lessdot n$ for any $n > 0$.

mar.5 Computable Models of Arithmetic

explanation

The standard model $\mathfrak{N}$ has two nice features. Its domain is the natural numbers $\mathbb{N}$, i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral $\overline{n}$ actually picks out n. The other nice feature is that the interpretations of the non-logical symbols of $\mathcal{L}_A$ are all *computable*. The successor, addition, and multiplication functions which serve as $\prime^{\mathfrak{N}}$, $+^{\mathfrak{N}}$, and $\times^{\mathfrak{N}}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on $\mathfrak{N}$, i.e., $<^{\mathfrak{N}}$, is decidable.

Non-standard models of arithmetical theories such as $\mathbf{Q}$ and $\mathbf{PA}$ must contain non-standard elements. Thus their domains typically include elements in addition to $\mathbb{N}$. However, any countable structure can be built on any denumerable set, including $\mathbb{N}$. So there are also non-standard models with domain $\mathbb{N}$. In such models $\mathfrak{M}$, of course, at least some numbers cannot play the roles they usually play, since some k must be different from $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all $n \in \mathbb{N}$.

Definition mar.10. A structure $\mathfrak{M}$ for $\mathcal{L}_A$ is *computable* iff $|\mathfrak{M}| = \mathbb{N}$ and $\prime^{\mathfrak{M}}$, $+^{\mathfrak{M}}$, $\times^{\mathfrak{M}}$ are computable functions and $<^{\mathfrak{M}}$ is a decidable relation.

Example mar.11. Recall the structure $\mathfrak{K}$ from Example mar.8 Its domain was $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$$\begin{aligned}
\mathrm{o}^{\mathfrak{K}} &= 0 \\
\prime^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\
+^{\mathfrak{K}}(x,y) &= \begin{cases} x+y & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
\times^{\mathfrak{K}}(x,y) &= \begin{cases} xy & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
<^{\mathfrak{K}} &= \{\langle x, y\rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a\rangle : n \in |\mathfrak{K}|\}
\end{aligned}$$

But $|\mathfrak{K}|$ is denumerable and so is equinumerous with $\mathbb{N}$. For instance, $g\colon \mathbb{N} \to |\mathfrak{K}|$ with $g(0) = a$ and $g(n) = n+1$ for $n > 0$ is a bijection. We can turn it into an isomorphism between a new model $\mathfrak{K}'$ of $\mathbf{Q}$ and $\mathfrak{K}$. In $\mathfrak{K}'$, we have to assign different functions and relations to the symbols of $\mathcal{L}_A$, since different elements of $\mathbb{N}$ play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of a, not of the smallest standard number. The smallest standard number is now 1. So we assign $\mathrm{o}^{\mathfrak{K}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n+1$. But 0 now plays the role of a, which is its own successor. So $\prime^{\mathfrak{K}'}(0) = 0$. For addition and multiplication we likewise have

$$\begin{aligned}
+^{\mathfrak{K}'}(x,y) &= \begin{cases} x+y & \text{if } x,\, y > 0 \\ 0 & \text{otherwise} \end{cases} \\
\times^{\mathfrak{K}'}(x,y) &= \begin{cases} xy & \text{if } x,\, y > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

And we have $\langle x, y\rangle \in <^{\mathfrak{K}'}$ iff $x < y$ and $x > 0$ and $y > 0$, or if $y = 0$.

All of these functions are computable functions of natural numbers and $<^{\mathfrak{K}'}$ is a decidable relation on $\mathbb{N}$—but they are not the same functions as successor, addition, and multiplication on $\mathbb{N}$, and $<^{\mathfrak{K}'}$ is not the same relation as $<$ on $\mathbb{N}$.

Problem mar.6. Give a structure $\mathfrak{L}'$ with $|\mathfrak{L}'| = \mathbb{N}$ isomorphic to $\mathfrak{L}$ of Example mar.9.

This example shows that $\mathbf{Q}$ has computable non-standard models with domain $\mathbb{N}$. However, the following result shows that this is not true for models of $\mathbf{PA}$ (and thus also for models of $\mathbf{TA}$). explanation

Theorem mar.12 (Tennenbaum's Theorem)**.** *$\mathfrak{N}$ is the only computable model of* $\mathbf{PA}$.

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Bibliography