

## Chapter udf

# Models of Arithmetic

### mar.1 Introduction

The *standard model* of arithmetic is the **structure**  $\mathfrak{N}$  with  $|\mathfrak{N}| = \mathbb{N}$  in which  $o$ ,  $l$ ,  $+$ ,  $\times$ , and  $<$  are interpreted as you would expect. That is,  $o$  is 0,  $l$  is the successor function,  $+$  is interpreted as addition and  $\times$  as multiplication of the numbers in  $\mathbb{N}$ . Specifically,

$$\begin{aligned}o^{\mathfrak{N}} &= 0 \\l^{\mathfrak{N}}(n) &= n + 1 \\+^{\mathfrak{N}}(n, m) &= n + m \\\times^{\mathfrak{N}}(n, m) &= nm\end{aligned}$$

Of course, there are structures for  $\mathcal{L}_A$  that have domains other than  $\mathbb{N}$ . For instance, we can take  $\mathfrak{M}$  with domain  $|\mathfrak{M}| = \{a\}^*$  (the finite sequences of the single symbol  $a$ , i.e.,  $\emptyset, a, aa, aaa, \dots$ ), and interpretations

$$\begin{aligned}o^{\mathfrak{M}} &= \emptyset \\l^{\mathfrak{M}}(s) &= s \frown a \\+^{\mathfrak{M}}(n, m) &= a^{n+m} \\\times^{\mathfrak{M}}(n, m) &= a^{nm}\end{aligned}$$

These two structures are “essentially the same” in the sense that the only difference is the **elements** of the **domains** but not how the **elements** of the **domains** are related among each other by the interpretation functions. We say that the two **structures** are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in  $\mathfrak{N}$  also has models that are not isomorphic to  $\mathfrak{N}$ . Such structures are called *non-standard*. The interesting thing about them is that while the **elements** of a standard model (i.e.,  $\mathfrak{N}$ , but also all **structures** isomorphic to it) are exhausted by the values of the standard numerals  $\bar{n}$ , i.e.,

$$|\mathfrak{N}| = \{\text{Val}^{\mathfrak{N}}(\bar{n}) : n \in \mathbb{N}\}$$

that isn't the case in non-standard models: if  $\mathfrak{M}$  is non-standard, then there is at least one  $x \in |\mathfrak{M}|$  such that  $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$  for all  $n$ .

These non-standard elements are pretty neat: they are “infinite natural numbers.” But their existence also explains, in a sense, the incompleteness phenomena. Consider an example, e.g., the consistency statement for Peano arithmetic,  $\text{Con}_{\mathbf{PA}}$ , i.e.,  $\neg \exists x \text{Prf}_{\mathbf{PA}}(x, \ulcorner \perp \urcorner)$ . Since  $\mathbf{PA}$  neither proves  $\text{Con}_{\mathbf{PA}}$  nor  $\neg \text{Con}_{\mathbf{PA}}$ , either can be consistently added to  $\mathbf{PA}$ . Since  $\mathbf{PA}$  is consistent,  $\mathfrak{N} \models \text{Con}_{\mathbf{PA}}$ , and consequently  $\mathfrak{N} \not\models \neg \text{Con}_{\mathbf{PA}}$ . So  $\mathfrak{N}$  is *not* a model of  $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$ , and all its models must be nonstandard. Models of  $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$  must contain some **element** that serves as the witness that makes  $\exists x \text{Prf}_{\mathbf{PA}}(\ulcorner \perp \urcorner)$  true, i.e., a Gödel number of a **derivation** of a contradiction from  $\mathbf{PA}$ . Such an **element** can't be standard—since  $\mathbf{PA} \vdash \neg \text{Prf}_{\mathbf{PA}}(\bar{n}, \ulcorner \perp \urcorner)$  for every  $n$ .

## mar.2 Standard Models of Arithmetic

The language of arithmetic  $\mathcal{L}_A$  is obviously intended to be about numbers, specifically, about natural numbers. So, “the” standard model  $\mathfrak{N}$  is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two **structures** that are isomorphic share those. So we can be a bit more liberal, and consider any **structure** that is isomorphic to  $\mathfrak{N}$  “standard.”

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**Definition mar.1.** A **structure** for  $\mathcal{L}_A$  is *standard* if it is isomorphic to  $\mathfrak{N}$ .

**Proposition mar.2.** *If a structure  $\mathfrak{M}$  is standard, then its domain is the set of values of the standard numerals, i.e.,*

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prop:standard-domain

$$|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$$

*Proof.* Clearly, every  $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$ . We just have to show that every  $x \in |\mathfrak{M}|$  is equal to  $\text{Val}^{\mathfrak{M}}(\bar{n})$  for some  $n$ . Since  $\mathfrak{M}$  is standard, it is isomorphic to  $\mathfrak{N}$ . Suppose  $g: \mathbb{N} \rightarrow |\mathfrak{M}|$  is an isomorphism. Then  $g(n) = g(\text{Val}^{\mathfrak{N}}(\bar{n})) = \text{Val}^{\mathfrak{M}}(\bar{n})$ . But for every  $x \in |\mathfrak{M}|$ , there is an  $n \in \mathbb{N}$  such that  $g(n) = x$ , since  $g$  is **surjective**.  $\square$

explanation

If a structure  $\mathfrak{M}$  for  $\mathcal{L}_A$  is standard, the elements of its **domain** can all be named by the standard numerals  $\bar{0}, \bar{1}, \bar{2}, \dots$ , i.e., the terms  $o, o', o''$ , etc. Of course, this does not mean that the **elements** of  $|\mathfrak{M}|$  are the numbers, just that we can pick them out the same way we can pick out the numbers in  $|\mathfrak{N}|$ .

**Problem mar.1.** Show that the converse of **Proposition mar.2** is false, i.e., give an example of a **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$  that is not isomorphic to  $\mathfrak{N}$ .

**Proposition mar.3.** *If  $\mathfrak{M} \models \mathbf{Q}$ , and  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ , then  $\mathfrak{M}$  is standard.*

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prop:thq-standard

*Proof.* We have to show that  $\mathfrak{M}$  is isomorphic to  $\mathfrak{N}$ . Consider the function  $g: \mathbb{N} \rightarrow |\mathfrak{M}|$  defined by  $g(n) = \text{Val}^{\mathfrak{M}}(\bar{n})$ . By the hypothesis,  $g$  is **surjective**. It is also **injective**:  $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$  whenever  $n \neq m$ . Thus, since  $\mathfrak{M} \models \mathbf{Q}$ ,  $\mathfrak{M} \models \bar{n} \neq \bar{m}$ , whenever  $n \neq m$ . Thus, if  $n \neq m$ , then  $\text{Val}^{\mathfrak{M}}(\bar{n}) \neq \text{Val}^{\mathfrak{M}}(\bar{m})$ , i.e.,  $g(n) \neq g(m)$ .

We also have to verify that  $g$  is an isomorphism.

1. We have  $g(o^{\mathfrak{N}}) = g(0)$  since,  $o^{\mathfrak{N}} = 0$ . By definition of  $g$ ,  $g(0) = \text{Val}^{\mathfrak{M}}(\bar{0})$ . But  $\bar{0}$  is just  $o$ , and the value of a term which happens to be a **constant symbol** is given by what the **structure** assigns to that **constant symbol**, i.e.,  $\text{Val}^{\mathfrak{M}}(o) = o^{\mathfrak{M}}$ . So we have  $g(o^{\mathfrak{N}}) = o^{\mathfrak{M}}$  as required.
2.  $g(\iota^{\mathfrak{N}}(n)) = g(n+1)$ , since  $\iota$  in  $\mathfrak{N}$  is the successor function on  $\mathbb{N}$ . Then,  $g(n+1) = \text{Val}^{\mathfrak{M}}(\overline{n+1})$  by definition of  $g$ . But  $\overline{n+1}$  is the same term as  $\bar{n}'$ , so  $\text{Val}^{\mathfrak{M}}(\overline{n+1}) = \text{Val}^{\mathfrak{M}}(\bar{n}')$ . By the definition of the value function, this is  $= \iota^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n}))$ . Since  $\text{Val}^{\mathfrak{M}}(\bar{n}) = g(n)$  we get  $g(\iota^{\mathfrak{N}}(n)) = \iota^{\mathfrak{M}}(g(n))$ .
3.  $g(+^{\mathfrak{N}}(n, m)) = g(n+m)$ , since  $+$  in  $\mathfrak{N}$  is the addition function on  $\mathbb{N}$ . Then,  $g(n+m) = \text{Val}^{\mathfrak{M}}(\overline{n+m})$  by definition of  $g$ . But  $\mathbf{Q} \vdash \bar{n} + \bar{m} = \overline{(n+m)}$ , so  $\text{Val}^{\mathfrak{M}}(\overline{n+m}) = \text{Val}^{\mathfrak{M}}(\bar{n} + \bar{m})$ . By the definition of the value function, this is  $= +^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n}), \text{Val}^{\mathfrak{M}}(\bar{m}))$ . Since  $\text{Val}^{\mathfrak{M}}(\bar{n}) = g(n)$  and  $\text{Val}^{\mathfrak{M}}(\bar{m}) = g(m)$ , we get  $g(+^{\mathfrak{N}}(n, m)) = +^{\mathfrak{M}}(g(n), g(m))$ .
4.  $g(\times^{\mathfrak{N}}(n, m)) = \times^{\mathfrak{M}}(g(n), g(m))$ : Exercise.
5.  $\langle n, m \rangle \in <^{\mathfrak{N}}$  iff  $n < m$ . If  $n < m$ , then  $\mathbf{Q} \vdash \bar{n} < \bar{m}$ , and also  $\mathfrak{M} \models \bar{n} < \bar{m}$ . Thus  $\langle \text{Val}^{\mathfrak{M}}(\bar{n}), \text{Val}^{\mathfrak{M}}(\bar{m}) \rangle \in <^{\mathfrak{M}}$ , i.e.,  $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$ . If  $n \not< m$ , then  $\mathbf{Q} \vdash \neg \bar{n} < \bar{m}$ , and consequently  $\mathfrak{M} \not\models \bar{n} < \bar{m}$ . Thus, as before,  $\langle g(n), g(m) \rangle \notin <^{\mathfrak{M}}$ . Together, we get:  $\langle n, m \rangle \in <^{\mathfrak{N}}$  iff  $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$ .  $\square$

The function  $g$  is the most obvious way of defining a mapping from  $\mathbb{N}$  to the domain of any other **structure**  $\mathfrak{M}$  for  $\mathcal{L}_A$ , since every such  $\mathfrak{M}$  contains **elements** named by  $\bar{0}$ ,  $\bar{1}$ ,  $\bar{2}$ , etc. So it isn't surprising that if  $\mathfrak{M}$  makes at least some basic statements about the  $\bar{n}$ 's true in the same way that  $\mathfrak{N}$  does, and  $g$  is also bijective, then  $g$  will turn into an isomorphism. In fact, if  $|\mathfrak{M}|$  contains no **elements** other than what the  $\bar{n}$ 's name, it's the only one. explanation

mod:mar:stm: prop:thq-unique-iso **Proposition mar.4.** *If  $\mathfrak{M}$  is standard, then  $g$  from the proof of **Proposition mar.3** is the only isomorphism from  $\mathfrak{N}$  to  $\mathfrak{M}$ .*

*Proof.* Suppose  $h: \mathbb{N} \rightarrow |\mathfrak{M}|$  is an isomorphism between  $\mathfrak{N}$  and  $\mathfrak{M}$ . We show that  $g = h$  by induction on  $n$ . If  $n = 0$ , then  $g(0) = o^{\mathfrak{M}}$  by definition of  $g$ . But since  $h$  is an isomorphism,  $h(0) = h(o^{\mathfrak{N}}) = o^{\mathfrak{M}}$ , so  $g(0) = h(0)$ .

Now consider the case for  $n + 1$ . We have

$$\begin{aligned}
g(n + 1) &= \text{Val}^{\mathfrak{M}}(\overline{n + 1}) \text{ by definition of } g \\
&= \text{Val}^{\mathfrak{M}}(\overline{n'}) \text{ since } \overline{n + 1} \equiv \overline{n'} \\
&= \iota^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\overline{n})) \text{ by definition of } \text{Val}^{\mathfrak{M}}(t') \\
&= \iota^{\mathfrak{M}}(g(n)) \text{ by definition of } g \\
&= \iota^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis} \\
&= h(\iota^{\mathfrak{N}}(n)) \text{ since } h \text{ is an isomorphism} \\
&= h(n + 1)
\end{aligned}$$

□

**explanation** For any **denumerable** set  $M$ , there's a **bijection** between  $\mathbb{N}$  and  $M$ , so every such set  $M$  is potentially the **domain** of a standard model  $\mathfrak{M}$ . In fact, once you pick an object  $z \in M$  and a suitable function  $s$  as  $\circ^{\mathfrak{M}}$  and  $\iota^{\mathfrak{M}}$ , the interpretations of  $+$ ,  $\times$ , and  $<$  is already fixed. Only functions  $s: M \rightarrow M \setminus \{z\}$  that are both **injective** and **surjective** are suitable in a standard model as  $\iota^{\mathfrak{M}}$ . The range of  $s$  cannot contain  $z$ , since otherwise  $\forall x \circ \neq x'$  would be false. That **sentence** is true in  $\mathfrak{N}$ , and so  $\mathfrak{M}$  also has to make it true. The function  $s$  has to be **injective**, since the successor function  $\iota^{\mathfrak{N}}$  in  $\mathfrak{N}$  is, and that  $\iota^{\mathfrak{M}}$  is **injective** is expressed by a **sentence** true in  $\mathfrak{N}$ . It has to be **surjective** because otherwise there would be some  $x \in M \setminus \{z\}$  not in the domain of  $s$ , i.e., the **sentence**  $\forall x (x = \circ \vee \exists y y' = x)$  would be false in  $\mathfrak{M}$ —but it is true in  $\mathfrak{N}$ .

### mar.3 Non-Standard Models

**explanation** We call a **structure** for  $\mathcal{L}_A$  standard if it is isomorphic to  $\mathfrak{N}$ . If a **structure** isn't isomorphic to  $\mathfrak{N}$ , it is called non-standard.

**Definition mar.5.** A **structure**  $\mathfrak{M}$  for  $\mathcal{L}_A$  is *non-standard* if it is not isomorphic to  $\mathfrak{N}$ . The **elements**  $x \in |\mathfrak{M}|$  which are equal to  $\text{Val}^{\mathfrak{M}}(\overline{n})$  for some  $n \in \mathbb{N}$  are called *standard numbers* (of  $\mathfrak{M}$ ), and those not, *non-standard numbers*.

**explanation** By **Proposition mar.2**, any standard **structure** for  $\mathcal{L}_A$  contains only standard **elements**. Consequently, a non-standard **structure** must contain at least one non-standard element. In fact, the existence of a non-standard **element** guarantees that the **structure** is non-standard.

**Proposition mar.6.** *If a structure  $\mathfrak{M}$  for  $\mathcal{L}_A$  contains a non-standard number,  $\mathfrak{M}$  is non-standard.*

*Proof.* Suppose not, i.e., suppose  $\mathfrak{M}$  standard but contains a non-standard number  $x$ . Let  $g: \mathbb{N} \rightarrow |\mathfrak{M}|$  be an isomorphism. It is easy to see (by induction on  $n$ ) that  $g(\text{Val}^{\mathfrak{N}}(\overline{n})) = \text{Val}^{\mathfrak{M}}(\overline{n})$ . In other words,  $g$  maps standard numbers of  $\mathfrak{N}$  to standard numbers of  $\mathfrak{M}$ . If  $\mathfrak{M}$  contains a non-standard number,  $g$  cannot be **surjective**, contrary to hypothesis. □

**Problem mar.2.** Recall that  $\mathbf{Q}$  contains the axioms

$$\forall x \forall y (x' = y' \rightarrow x = y) \quad (Q_1)$$

$$\forall x \ 0 \neq x' \quad (Q_2)$$

$$\forall x (x = 0 \vee \exists y x = y') \quad (Q_3)$$

Give **structures**  $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$  such that

1.  $\mathfrak{M}_1 \models Q_1, \mathfrak{M}_1 \models Q_2, \mathfrak{M}_1 \not\models Q_3$ ;
2.  $\mathfrak{M}_2 \models Q_1, \mathfrak{M}_2 \not\models Q_2, \mathfrak{M}_2 \models Q_3$ ; and
3.  $\mathfrak{M}_3 \not\models Q_1, \mathfrak{M}_3 \models Q_2, \mathfrak{M}_3 \models Q_3$ ;

Obviously, you just have to specify  $0^{\mathfrak{M}_i}$  and  $'^{\mathfrak{M}_i}$  for each.

It is easy enough to specify non-standard **structures** for  $\mathcal{L}_A$ . For instance, [explanation](#) take the structure with **domain**  $\mathbb{Z}$  and interpret all non-logical symbols as usual. Since negative numbers are not values of  $\bar{n}$  for any  $n$ , this structure is non-standard. Of course, it will not be a *model* of arithmetic in the sense that it makes the same sentences true as  $\mathfrak{N}$ . For instance,  $\forall x x' \neq 0$  is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

**Proposition mar.7.** *Let  $\mathbf{TA} = \{\varphi : \mathfrak{N} \models \varphi\}$  be the theory of  $\mathfrak{N}$ .  $\mathbf{TA}$  has an **enumerable non-standard model**.*

*Proof.* Expand  $\mathcal{L}_A$  by a new **constant symbol**  $c$  and consider the set of **sentences**

$$\Gamma = \mathbf{TA} \cup \{c \neq \bar{0}, c \neq \bar{1}, c \neq \bar{2}, \dots\}$$

Any model  $\mathfrak{M}^c$  of  $\Gamma$  would contain an **element**  $x = c^{\mathfrak{M}}$  which is non-standard, since  $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$  for all  $n \in \mathbb{N}$ . Also, obviously,  $\mathfrak{M}^c \models \mathbf{TA}$ , since  $\mathbf{TA} \subseteq \Gamma$ . If we turn  $\mathfrak{M}^c$  into a **structure**  $\mathfrak{M}$  for  $\mathcal{L}_A$  simply by forgetting about  $c$ , its domain still contains the non-standard  $x$ , and also  $\mathfrak{M} \models \mathbf{TA}$ . The latter is guaranteed since  $c$  does not occur in  $\mathbf{TA}$ . So, it suffices to show that  $\Gamma$  has a model.

We use the compactness theorem to show that  $\Gamma$  has a model. If every finite subset of  $\Gamma$  is satisfiable, so is  $\Gamma$ . Consider any finite subset  $\Gamma_0 \subseteq \Gamma$ .  $\Gamma_0$  includes some **sentences** of  $\mathbf{TA}$  and some of the form  $c \neq \bar{k}$ , but only finitely many. Suppose  $k$  is the largest number so that  $c \neq \bar{k} \in \Gamma_0$ . Define  $\mathfrak{N}_k$  by expanding  $\mathfrak{N}$  to include the interpretation  $c^{\mathfrak{N}_k} = k + 1$ .  $\mathfrak{N}_k \models \Gamma_0$ : if  $\varphi \in \mathbf{TA}$ ,  $\mathfrak{N}_k \models \varphi$  since  $\mathfrak{N}_k$  is just like  $\mathfrak{N}$  in all respects except  $c$ , and  $c$  does not occur in  $\varphi$ . And  $\mathfrak{N}_k \models c \neq \bar{n}$ , since  $n \leq k$ , and  $\text{Val}^{\mathfrak{N}_k}(c) = k + 1$ . Thus, every finite subset of  $\Gamma$  is satisfiable.  $\square$

## mar.4 Models of $\mathbf{Q}$

explanation We know that there are non-standard **structures** that make the same **sentences** true as  $\mathfrak{N}$  does, i.e., is a model of **TA**. Since  $\mathfrak{N} \models \mathbf{Q}$ , any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g.,  $\mathbf{Q} \not\models \forall x \forall y (x + y) = (y + x)$ . Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make  $\forall x \forall y (x + y) = (y + x)$  false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly.

**Example mar.8.** Consider the **structure**  $\mathfrak{K}$  with domain  $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$  and interpretations

mod:mar:mdq:  
ex:model-K-of-Q

$$\begin{aligned} 0^{\mathfrak{K}} &= 0 \\ \iota^{\mathfrak{K}}(x) &= \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

To show that  $\mathfrak{K} \models \mathbf{Q}$  we have to verify that all axioms of **Q** are true in  $\mathfrak{K}$ . For convenience, let's write  $x^*$  for  $\iota^{\mathfrak{K}}(x)$  (the “successor” of  $x$  in  $\mathfrak{K}$ ),  $x \oplus y$  for  $+^{\mathfrak{K}}(x, y)$  (the “sum” of  $x$  and  $y$  in  $\mathfrak{K}$ ),  $x \otimes y$  for  $\times^{\mathfrak{K}}(x, y)$  (the “product” of  $x$  and  $y$  in  $\mathfrak{K}$ ), and  $x \odot y$  for  $\langle x, y \rangle \in <^{\mathfrak{K}}$ . With these abbreviations, we can give the operations in  $\mathfrak{K}$  more perspicuously as

$x$	$x^*$	$x \oplus y$	0	$m$	$a$	$x \otimes y$	0	$m$	$a$
$n$	$n + 1$	0	0	$m$	$a$	0	0	0	0
$a$	$a$	$n$	$n$	$n + m$	$a$	$n$	0	$nm$	$a$
		$a$	$a$	$a$	$a$	$a$	0	$a$	$a$

We have  $n \odot m$  iff  $n < m$  for  $n, m \in \mathbb{N}$  and  $x \odot a$  for all  $x \in |\mathfrak{K}|$ .

$\mathfrak{K} \models \forall x \forall y (x' = y' \rightarrow x = y)$  since  $*$  is **injective**.  $\mathfrak{K} \models \forall x 0 \neq x'$  since 0 is not a  $*$ -successor in  $\mathfrak{K}$ .  $\mathfrak{K} \models \forall x (x = 0 \vee \exists y x = y')$  since for every  $n > 0$ ,  $n = (n - 1)^*$ , and  $a = a^*$ .

$\mathfrak{K} \models \forall x (x \oplus 0) = x$  since  $n \oplus 0 = n + 0 = n$ , and  $a \oplus 0 = a$  by definition of  $\oplus$ .  $\mathfrak{K} \models \forall x \forall y (x \oplus y') = (x + y)'$  is a bit trickier. If  $n, m$  are both standard, we have:

$$(n \oplus m^*) = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*$$

since  $\oplus$  and  $*$  agree with  $+$  and  $!$  on standard numbers. Now suppose  $x \in |\mathfrak{K}|$ . Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if  $y \in |\mathfrak{K}|$  but  $x = a$ . Here we also have to distinguish cases according to whether  $y = n$  is standard or  $y = b$ :

$$(a \oplus n^*) = (a \oplus (n + 1)) = a = a^* = (a \oplus n)^*$$

$$(a \oplus a^*) = (a \oplus a) = a = a^* = (a \oplus a)^*$$

This is of course a bit more detailed than needed. For instance, since  $a \oplus z = a$  whatever  $z$  is, we can immediately conclude  $a \oplus a^* = a$ . The remaining axioms can be verified the same way.

$\mathfrak{K}$  is thus a model of  $\mathbf{Q}$ . Its “addition”  $\oplus$  is also commutative. But there are other sentences true in  $\mathfrak{N}$  but false in  $\mathfrak{K}$ , and vice versa. For instance,  $a \otimes a$ , so  $\mathfrak{K} \models \exists x x < x$  and  $\mathfrak{K} \not\models \forall x \neg x < x$ . This shows that  $\mathbf{Q} \not\models \forall x \neg x < x$ .

**Problem mar.3.** Prove that  $\mathfrak{K}$  from [Example mar.8](#) satisfies the remaining axioms of  $\mathbf{Q}$ ,

$$\forall x (x \times 0) = 0 \tag{Q_6}$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \tag{Q_7}$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (z' + x) = y) \tag{Q_8}$$

Find a sentence only involving  $!$  true in  $\mathfrak{N}$  but false in  $\mathfrak{K}$ .

[mod:mar:mdq:](#)  
[ex:model-L-of-Q](#)

**Example mar.9.** Consider the structure  $\mathfrak{L}$  with domain  $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$  and interpretations  $!^{\mathfrak{L}} = *$ ,  $+^{\mathfrak{L}} = \oplus$  given by

$x$	$x^*$	$x \oplus y$	$m$	$a$	$b$
$n$	$n + 1$	$n$	$n + m$	$b$	$a$
$a$	$a$	$a$	$a$	$b$	$a$
$b$	$b$	$b$	$b$	$b$	$a$

Since  $*$  is injective, 0 is not in its range, and every  $x \in |\mathfrak{L}|$  other than 0 is, axioms  $Q_1$ – $Q_3$  are true in  $\mathfrak{L}$ . For any  $x$ ,  $x \oplus 0 = x$ , so  $Q_4$  is true as well. For  $Q_5$ , consider  $x \oplus y^*$  and  $(x \oplus y)^*$ . They are equal if  $x$  and  $y$  are both standard, since then  $*$  and  $\oplus$  agree with  $!$  and  $+$ . If  $x$  is non-standard, and  $y$  is standard, we have  $x \oplus y^* = x = x^* = (x \oplus y)^*$ . If  $x$  and  $y$  are both non-standard, we have four cases:

$$a \oplus a^* = b = b^* = (a \oplus a)^*$$

$$b \oplus b^* = a = a^* = (b \oplus b)^*$$

$$b \oplus a^* = b = b^* = (b \oplus y)^*$$

$$a \oplus b^* = a = a^* = (a \oplus b)^*$$

If  $x$  is standard, but  $y$  is non-standard, we have

$$\begin{aligned} n \oplus a^* &= n \oplus a = b = b^* = (n \oplus a)^* \\ n \oplus b^* &= n \oplus b = a = a^* = (n \oplus b)^* \end{aligned}$$

So,  $\mathcal{L} \models Q_5$ . However,  $a \oplus 0 \neq 0 \oplus a$ , so  $\mathcal{L} \not\models \forall x \forall y (x + y) = (y + x)$ .

**Problem mar.4.** Expand  $\mathcal{L}$  of [Example mar.9](#) to include  $\otimes$  and  $\ominus$  that interpret  $\times$  and  $<$ . Show that your structure satisfies the remaining axioms of  $\mathbf{Q}$ ,

$$\forall x (x \times 0) = 0 \tag{Q_6}$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \tag{Q_7}$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (z' + x) = y) \tag{Q_8}$$

**Problem mar.5.** In  $\mathcal{L}$  of [Example mar.9](#),  $a^* = a$  and  $b^* = b$ . Is there a model of  $\mathbf{Q}$  in which  $a^* = b$  and  $b^* = a$ ?

explanation

We've explicitly constructed models of  $\mathbf{Q}$  in which the non-standard [elements](#) live “beyond” the standard elements. In fact, that much is required by the axioms. A non-standard [element](#)  $x$  cannot be  $\ominus 0$ , since  $\mathbf{Q} \vdash \forall x \neg x < 0$  (see ??). Also, for every  $n$ ,  $\mathbf{Q} \vdash \forall x (x < \bar{n}' \rightarrow (x = \bar{0} \vee x = \bar{1} \vee \dots \vee x = \bar{n}))$  (??), so we can't have  $a \ominus n$  for any  $n > 0$ .

## mar.5 Models of PA

explanation

Any non-standard model of  $\mathbf{TA}$  is also one of  $\mathbf{PA}$ . We know that non-standard models of  $\mathbf{TA}$  and hence of  $\mathbf{PA}$  exist. We also know that such non-standard models contain non-standard “numbers,” i.e., [elements](#) of the domain that are “beyond” all the standard “numbers.” But how are they arranged? How many are there? We've seen that models of the weaker theory  $\mathbf{Q}$  can contain as few as a single non-standard number. But these simple [structures](#) are not models of  $\mathbf{PA}$  or  $\mathbf{TA}$ .

The key to understanding the structure of models of  $\mathbf{PA}$  or  $\mathbf{TA}$  is to see what facts are [derivable](#) in these theories. For instance, already  $\mathbf{PA}$  proves that  $\forall x x \neq x'$  and  $\forall x \forall y (x + y) = (y + x)$ , so this rules out simple structures (in which these [sentences](#) are false) as models of  $\mathbf{PA}$ .

Suppose  $\mathfrak{M}$  is a model of  $\mathbf{PA}$ . Then if  $\mathbf{PA} \vdash \varphi$ ,  $\mathfrak{M} \models \varphi$ . Let's again use  $\mathbf{z}$  for  $0^{\mathfrak{M}}$ ,  $*$  for  $1^{\mathfrak{M}}$ ,  $\oplus$  for  $+^{\mathfrak{M}}$ ,  $\otimes$  for  $\times^{\mathfrak{M}}$ , and  $\ominus$  for  $<^{\mathfrak{M}}$ . Any [sentence](#)  $\varphi$  then states some condition about  $\mathbf{z}$ ,  $*$ ,  $\oplus$ ,  $\otimes$ , and  $\ominus$ , and if  $\mathfrak{M} \models \varphi$  that condition must be satisfied. For instance, if  $\mathfrak{M} \models Q_1$ , i.e.,  $\mathfrak{M} \models \forall x \forall y (x' = y' \rightarrow x = y)$ , then  $*$  must be [injective](#).

**Proposition mar.10.** In  $\mathfrak{M}$ ,  $\ominus$  is a linear strict order, i.e., it satisfies:

1. Not  $x \ominus x$  for any  $x \in |\mathfrak{M}|$ .



2. If  $x \otimes y$  and  $y \otimes z$  then  $x \otimes z$ .
3. For any  $x \neq y$ ,  $x \otimes y$  or  $y \otimes x$

*Proof.* **PA** proves:

1.  $\forall x \neg x < x$
2.  $\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$
3.  $\forall x \forall y ((x < y \vee y < x) \vee x = y)$  □

*mod:mar:mpa:  
prop:M-discrete*

**Proposition mar.11.**  $\mathbf{z}$  is the least *element* of  $|\mathfrak{M}|$  in the  $\otimes$ -ordering. For any  $x$ ,  $x \otimes x^*$ , and  $x^*$  is the  $\otimes$ -least *element* with that property. For any  $x$ , there is a unique  $y$  such that  $y^* = x$ . (We call  $y$  the “predecessor” of  $x$  in  $\mathfrak{M}$ , and denote it by  ${}^*x$ .)

*Proof.* Exercise. □

**Problem mar.6.** Find *sentences* in  $\mathcal{L}_A$  *derivable* in **PA** (and hence true in  $\mathfrak{N}$ ) which guarantee the properties of  $\mathbf{z}$ ,  $*$ , and  $\otimes$  in **Proposition mar.11**

**Proposition mar.12.** All standard *elements* of  $\mathfrak{M}$  are less than (according to  $\otimes$ ) all non-standard *elements*.

*Proof.* We’ll use  $n$  as short for  $\text{Val}^{\mathfrak{M}}(\bar{n})$ , a standard *element* of  $\mathfrak{M}$ . Already **Q** proves that, for any  $n \in \mathbb{N}$ ,  $\forall x (x < \bar{n}' \rightarrow (x = \bar{0} \vee x = \bar{1} \vee \dots \vee x = \bar{n}))$ . There are no *elements* that are  $\otimes \mathbf{z}$ . So if  $n$  is standard and  $x$  is non-standard, we cannot have  $x \otimes n$ . By definition, a non-standard element is one that isn’t  $\text{Val}^{\mathfrak{M}}(\bar{n})$  for any  $n \in \mathbb{N}$ , so  $x \neq n$  as well. Since  $\otimes$  is a linear order, we must have  $n \otimes x$ . □

**Proposition mar.13.** Every nonstandard *element*  $x$  of  $|\mathfrak{M}|$  is an element of the subset

$$\dots {}^{***}x \otimes {}^{**}x \otimes {}^*x \otimes x \otimes x \otimes x^* \otimes x^{**} \otimes x^{***} \otimes \dots$$

We call this subset the *block* of  $x$  and write it as  $[x]$ . It has no least and no greatest *element*. It can be characterized as the set of those  $y \in |\mathfrak{M}|$  such that, for some standard  $n$ ,  $x \oplus n = y$  or  $y \oplus n = x$ .

*Proof.* Clearly, such a set  $[x]$  always exists since every *element*  $y$  of  $|\mathfrak{M}|$  has a unique successor  $y^*$  and unique predecessor  ${}^*y$ . For successive *elements*  $y$ ,  $y^*$  we have  $y \otimes y^*$  and  $y^*$  is the  $\otimes$ -least *element* of  $|\mathfrak{M}|$  such that  $y$  is  $\otimes$ -less than it. Since always  ${}^*y \otimes y$  and  $y \otimes y^*$ ,  $[x]$  has no least or greatest *element*. If  $y \in [x]$  then  $x \in [y]$ , for then either  $y^{* \dots *} = x$  or  $x^{* \dots *} = y$ . If  $y^{* \dots *} = x$  (with  $n$   $*$ ’s), then  $y \oplus n = x$  and conversely, since **PA**  $\vdash \forall x x' \dots' = (x + \bar{n})$  (if  $n$  is the number of  $\prime$ ’s). □

**Proposition mar.14.** *If  $[x] \neq [y]$  and  $x \otimes y$ , then for any  $u \in [x]$  and any  $v \in [y]$ ,  $u \otimes v$ .*

*Proof.* Note that  $\mathbf{PA} \vdash \forall x \forall y (x < y \rightarrow (x' < y \vee x' = y))$ . Thus, if  $u \otimes v$ , we also have  $u \oplus n^* \otimes v$  for any  $n$  if  $[u] \neq [v]$ .

Any  $u \in [x]$  is  $\otimes y$ :  $x \otimes y$  by assumption. If  $u \otimes x$ ,  $u \otimes y$  by transitivity. And if  $x \otimes u$  but  $u \in [x]$ , we have  $u = x \oplus n^*$  for some  $n$ , and so  $u \otimes y$  by the fact just proved.

Now suppose that  $v \in [y]$  is  $\otimes y$ , i.e.,  $v \oplus m^* = y$  for some standard  $m$ . This rules out  $v \otimes x$ , otherwise  $y = v \oplus m^* \otimes x$ . Clearly also,  $x \neq v$ , otherwise  $x \oplus m^* = v \oplus m^* = y$  and we would have  $[x] = [y]$ . So,  $x \otimes v$ . But then also  $x \oplus n^* \otimes v$  for any  $n$ . Hence, if  $x \otimes u$  and  $u \in [x]$ , we have  $u \otimes v$ . If  $u \otimes x$  then  $u \otimes v$  by transitivity.

Lastly, if  $y \otimes v$ ,  $u \otimes v$  since, as we've shown,  $u \otimes y$  and  $y \otimes v$ . □

**Corollary mar.15.** *If  $[x] \neq [y]$ ,  $[x] \cap [y] = \emptyset$ .*

*Proof.* Suppose  $z \in [x]$  and  $x \otimes y$ . Then  $z \otimes u$  for all  $u \in [y]$ . If  $z \in [y]$ , we would have  $z \otimes z$ . Similarly if  $y \otimes x$ . □

explanation

This means that the blocks themselves can be ordered in a way that respects  $\otimes$ :  $[x] \otimes [y]$  iff  $x \otimes y$ , or, equivalently, if  $u \otimes v$  for any  $u \in [x]$  and  $v \in [y]$ . Clearly, the standard block  $[0]$  is the least block. It intersects with no non-standard block, and no two non-standard blocks intersect either. Specifically, you cannot “reach” a different block by taking repeated successors or predecessors.

**Proposition mar.16.** *If  $x$  and  $y$  are non-standard, then  $x \otimes x \oplus y$  and  $x \oplus y \notin [x]$ .*

*Proof.* If  $y$  is nonstandard, then  $y \neq \mathbf{z}$ .  $\mathbf{PA} \vdash \forall x (y \neq \mathbf{0} \rightarrow x < (x + y))$ . Now suppose  $x \oplus y \in [x]$ . Since  $x \otimes x \oplus y$ , we would have  $x \oplus n^* = x \oplus y$ . But  $\mathbf{PA} \vdash \forall x \forall y \forall z ((x + y) = (x + z) \rightarrow y = z)$  (the cancellation law for addition). This would mean  $y = n^*$  for some standard  $n$ ; but  $y$  is assumed to be non-standard. □

**Proposition mar.17.** *There is no least non-standard block.*

*Proof.*  $\mathbf{PA} \vdash \forall x \exists y ((y + y) = x \vee (y + y)' = x)$ , i.e., that every  $x$  is divisible by 2 (possibly with remainder 1). If  $x$  is non-standard, so is  $y$ . By the preceding proposition,  $y \otimes y \oplus y$  and  $y \oplus y \notin [y]$ . Then also  $y \otimes (y \oplus y)^*$  and  $(y \oplus y)^* \notin [y]$ . But  $x = y \oplus y$  or  $x = (y \oplus y)^*$ , so  $y \otimes x$  and  $y \notin [x]$ . □

**Proposition mar.18.** *There is no largest block.*

*Proof.* Exercise. □

**Problem mar.7.** Show that in a non-standard model of  $\mathbf{PA}$ , there is no largest block.

*mod:mar:mpa:*  
*prop:blocks-dense*

**Proposition mar.19.** *The ordering of the blocks is dense. That is, if  $x \otimes y$  and  $[x] \neq [y]$ , then there is a block  $[z]$  distinct from both that is between them.*

*Proof.* Suppose  $x \otimes y$ . As before,  $x \oplus y$  is divisible by two (possibly with remainder): there is a  $z \in |\mathfrak{M}|$  such that either  $x \oplus y = z \oplus z$  or  $x \oplus y = (z \oplus z)^*$ . The element  $z$  is the “average” of  $x$  and  $y$ , and  $x \otimes z$  and  $z \otimes y$ .  $\square$

**Problem mar.8.** Write out a detailed proof of **Proposition mar.19**. Which sentence must **PA derive** in order to guarantee the existence of  $z$ ? Why is  $x \otimes z$  and  $z \otimes y$ , and why is  $[x] \neq [z]$  and  $[z] \neq [y]$ ?

The non-standard blocks are therefore ordered like the rationals: they form a **denumerable** dense linear ordering without endpoints. One can show that any two such **denumerable** orderings are isomorphic. It follows that for any two **enumerable** non-standard models  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  of true arithmetic, their reducts to the language containing  $<$  and  $=$  only are isomorphic. Indeed, an isomorphism  $h$  can be defined as follows: the standard parts of  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are isomorphic to the standard model  $\mathfrak{N}$  and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore isomorphic; an isomorphism of the blocks can be extended to an isomorphism *within* the blocks by matching up arbitrary elements in each, and then taking the image of the successor of  $x$  in  $\mathfrak{M}_1$  to be the successor of the image of  $x$  in  $\mathfrak{M}_2$ . Note that it does *not* follow that  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are isomorphic in the full language of arithmetic (indeed, isomorphism is always relative to a **language**), as there are non-isomorphic ways to define addition and multiplication over  $|\mathfrak{M}_1|$  and  $|\mathfrak{M}_2|$ . (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be 2.) explanation

## mar.6 Computable Models of Arithmetic

The standard model  $\mathfrak{N}$  has two nice features. Its domain is the natural numbers  $\mathbb{N}$ , i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral  $\bar{n}$  actually picks out  $n$ . The other nice feature is that the interpretations of the non-logical symbols of  $\mathcal{L}_A$  are all *computable*. The successor, addition, and multiplication functions which serve as  $s^{\mathfrak{N}}$ ,  $+$  <sup>$\mathfrak{N}$</sup> , and  $\times^{\mathfrak{N}}$  are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on  $\mathfrak{N}$ , i.e.,  $<^{\mathfrak{N}}$ , is decidable. explanation

Non-standard models of arithmetical theories such as **Q** and **PA** must contain non-standard elements. Thus their domains typically include **elements** in addition to  $\mathbb{N}$ . However, any countable **structure** can be built on any **denumerable** set, including  $\mathbb{N}$ . So there are also non-standard models with domain  $\mathbb{N}$ . In such models  $\mathfrak{M}$ , of course, at least some numbers cannot play the roles they usually play, since some  $k$  must be different from  $\text{Val}^{\mathfrak{M}}(\bar{n})$  for all  $n \in \mathbb{N}$ .

**Definition mar.20.** A structure  $\mathfrak{M}$  for  $\mathcal{L}_A$  is *computable* iff  $|\mathfrak{M}| = \mathbb{N}$  and  $r^{\mathfrak{M}}$ ,  $+^{\mathfrak{M}}$ ,  $\times^{\mathfrak{M}}$  are computable functions and  $<^{\mathfrak{M}}$  is a decidable relation.

**Example mar.21.** Recall the structure  $\mathfrak{K}$  from [Example mar.8](#). Its domain was  $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$  and interpretations

mod:mar:cmp:  
ex:comp-model-q

$$\begin{aligned} o^{\mathfrak{K}} &= 0 \\ r^{\mathfrak{K}}(x) &= \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

But  $|\mathfrak{K}|$  is *denumerable* and so is equinumerous with  $\mathbb{N}$ . For instance,  $g: \mathbb{N} \rightarrow |\mathfrak{K}|$  with  $g(0) = a$  and  $g(n) = n + 1$  for  $n > 0$  is a *bijection*. We can turn it into an isomorphism between a new model  $\mathfrak{K}'$  of  $\mathbf{Q}$  and  $\mathfrak{K}$ . In  $\mathfrak{K}'$ , we have to assign different functions and relations to the symbols of  $\mathcal{L}_A$ , since different *elements* of  $\mathbb{N}$  play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of  $a$ , not of the smallest standard number. The smallest standard number is now 1. So we assign  $o^{\mathfrak{K}'} = 1$ . The successor function is also different now: given a standard number, i.e., an  $n > 0$ , it still returns  $n + 1$ . But 0 now plays the role of  $a$ , which is its own successor. So  $r^{\mathfrak{K}'}(0) = 0$ . For addition and multiplication we likewise have

$$\begin{aligned} +^{\mathfrak{K}'}(x, y) &= \begin{cases} x + y - 1 & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}'}(x, y) &= \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ xy - x - y + 2 & \text{if } x, y > 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

And we have  $\langle x, y \rangle \in <^{\mathfrak{K}'}$  iff  $x < y$  and  $x > 0$  and  $y > 0$ , or if  $y = 0$ .

All of these functions are computable functions of natural numbers and  $<^{\mathfrak{K}'}$  is a decidable relation on  $\mathbb{N}$ —but they are not the same functions as successor, addition, and multiplication on  $\mathbb{N}$ , and  $<^{\mathfrak{K}'}$  is not the same relation as  $<$  on  $\mathbb{N}$ .

**Problem mar.9.** Give a *structure*  $\mathfrak{L}'$  with  $|\mathfrak{L}'| = \mathbb{N}$  isomorphic to  $\mathfrak{L}$  of [Example mar.9](#).

explanation

[Example mar.21](#) shows that  $\mathbf{Q}$  has computable non-standard models with domain  $\mathbb{N}$ . However, the following result shows that this is not true for models of  $\mathbf{PA}$  (and thus also for models of  $\mathbf{TA}$ ).

**Theorem mar.22 (Tennenbaum's Theorem).**  $\mathfrak{N}$  is the only computable model of PA.

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# Bibliography