

Chapter udf

Models of Arithmetic

mar.1 Introduction

The *standard model* of arithmetic is the **structure** \mathfrak{N} with $|\mathfrak{N}| = \mathbb{N}$ in which \circ , ι , $+$, \times , and $<$ are interpreted as you would expect. That is, \circ is 0, ι is the successor function, $+$ is interpreted as addition and \times as multiplication of the numbers in \mathbb{N} . Specifically,

$$\begin{aligned}\circ^{\mathfrak{N}} &= 0 \\ \iota^{\mathfrak{N}}(n) &= n + 1 \\ +^{\mathfrak{N}}(n, m) &= n + m \\ \times^{\mathfrak{N}}(n, m) &= nm\end{aligned}$$

Of course, there are structures for \mathcal{L}_A that have domains other than \mathbb{N} . For instance, we can take \mathfrak{M} with domain $|\mathfrak{M}| = \{a\}^*$ (the finite sequences of the single symbol a , i.e., $\emptyset, a, aa, aaa, \dots$), and interpretations

$$\begin{aligned}\circ^{\mathfrak{M}} &= \emptyset \\ \iota^{\mathfrak{M}}(s) &= s \frown a \\ +^{\mathfrak{M}}(n, m) &= a^{n+m} \\ \times^{\mathfrak{M}}(n, m) &= a^{nm}\end{aligned}$$

These two structures are “essentially the same” in the sense that the only difference is the **elements** of the **domains** but not how the **elements** of the **domains** are related among each other by the interpretation functions. We say that the two **structures** are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in \mathfrak{N} also has models that are not isomorphic to \mathfrak{N} . Such structures are called *non-standard*. The interesting thing about them is that while the **elements** of a standard model (i.e., \mathfrak{N} , but also all **structures** isomorphic to it) are exhausted by the values of the standard numerals \bar{n} , i.e.,

$$|\mathfrak{N}| = \{\text{Val}^{\mathfrak{N}}(\bar{n}) : n \in \mathbb{N}\}$$

that isn't the case in non-standard models: if \mathfrak{M} is non-standard, then there is at least one $x \in |\mathfrak{M}|$ such that $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$ for all n .

These non-standard elements are pretty neat: they are “infinite natural numbers.” But their existence also explains, in a sense, the incompleteness phenomena. Consider an example, e.g., the consistency statement for Peano arithmetic, $\text{Con}_{\mathbf{PA}}$, i.e., $\neg \exists x \text{Prf}_{\mathbf{PA}}(x, \ulcorner \perp \urcorner)$. Since \mathbf{PA} neither proves $\text{Con}_{\mathbf{PA}}$ nor $\neg \text{Con}_{\mathbf{PA}}$, either can be consistently added to \mathbf{PA} . Since \mathbf{PA} is consistent, $\mathfrak{N} \models \text{Con}_{\mathbf{PA}}$, and consequently $\mathfrak{N} \not\models \neg \text{Con}_{\mathbf{PA}}$. So \mathfrak{N} is *not* a model of $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$, and all its models must be nonstandard. Models of $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$ must contain some **element** that serves as the witness that makes $\exists x \text{Prf}_{\mathbf{PA}}(\ulcorner \perp \urcorner)$ true, i.e., a Gödel number of a **derivation** of a contradiction from \mathbf{PA} . Such an **element** can't be standard—since $\mathbf{PA} \vdash \neg \text{Prf}_{\mathbf{PA}}(\bar{n}, \ulcorner \perp \urcorner)$ for every n .

mar.2 Standard Models of Arithmetic

The language of arithmetic \mathcal{L}_A is obviously intended to be about numbers, specifically, about natural numbers. So, “the” standard model \mathfrak{N} is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two **structures** that are isomorphic share those. So we can be a bit more liberal, and consider any **structure** that is isomorphic to \mathfrak{N} “standard.”

mod:mar:stm:
sec

Definition mar.1. A **structure** for \mathcal{L}_A is *standard* if it is isomorphic to \mathfrak{N} .

Proposition mar.2. If a **structure** \mathfrak{M} is standard, its domain is the set of values of the standard numerals, i.e.,

mod:mar:stm:
prop:standard-domain

$$|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$$

Proof. Clearly, every $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$. We just have to show that every $x \in |\mathfrak{M}|$ is equal to $\text{Val}^{\mathfrak{M}}(\bar{n})$ for some n . Since \mathfrak{M} is standard, it is isomorphic to \mathfrak{N} . Suppose $g: \mathbb{N} \rightarrow |\mathfrak{M}|$ is an isomorphism. Then $g(n) = g(\text{Val}^{\mathfrak{N}}(\bar{n})) = \text{Val}^{\mathfrak{M}}(\bar{n})$. But for every $x \in |\mathfrak{M}|$, there is an $n \in \mathbb{N}$ such that $g(n) = x$, since g is **surjective**. \square

explanation

If a structure \mathfrak{M} for \mathcal{L}_A is standard, the elements of its **domain** can all be named by the standard numerals $\bar{0}, \bar{1}, \bar{2}, \dots$, i.e., the terms o, o', o'', \dots . Of course, this does not mean that the **elements** of $|\mathfrak{M}|$ are the numbers, just that we can pick them out the same way we can pick out the numbers in $|\mathfrak{N}|$.

Problem mar.1. Show that the converse of **Proposition mar.2** is false, i.e., give an example of a **structure** \mathfrak{M} with $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ that is not isomorphic to \mathfrak{N} .

Proposition mar.3. If $\mathfrak{M} \models \mathbf{Q}$, and $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$, then \mathfrak{M} is *standard*.

mod:mar:stm:
prop:thq-standard

Proof. We have to show that \mathfrak{M} is isomorphic to \mathfrak{N} . Consider the function $g: \mathbb{N} \rightarrow |\mathfrak{M}|$ defined by $g(n) = \text{Val}^{\mathfrak{M}}(\bar{n})$. By the hypothesis, g is **surjective**. It is also **injective**: $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$ whenever $n \neq m$. Thus, since $\mathfrak{M} \models \mathbf{Q}$, $\mathfrak{M} \models \bar{n} \neq \bar{m}$, whenever $n \neq m$. Thus, if $n \neq m$, then $\text{Val}^{\mathfrak{M}}(\bar{n}) \neq \text{Val}^{\mathfrak{M}}(\bar{m})$, i.e., $g(n) \neq g(m)$.

We also have to verify that g is an isomorphism.

1. We have $g(o^{\mathfrak{N}}) = g(0)$ since, $o^{\mathfrak{N}} = 0$. By definition of g , $g(0) = \text{Val}^{\mathfrak{M}}(\bar{0})$. But $\bar{0}$ is just o , and the value of a term which happens to be a **constant symbol** is given by what the **structure** assigns to that **constant symbol**, i.e., $\text{Val}^{\mathfrak{M}}(o) = o^{\mathfrak{M}}$. So we have $g(o^{\mathfrak{N}}) = o^{\mathfrak{M}}$ as required.
2. $g(\iota^{\mathfrak{N}}(n)) = g(n+1)$, since ι in \mathfrak{N} is the successor function on \mathbb{N} . Then, $g(n+1) = \text{Val}^{\mathfrak{M}}(\overline{n+1})$ by definition of g . But $\overline{n+1}$ is the same term as \bar{n}' , so $\text{Val}^{\mathfrak{M}}(\overline{n+1}) = \text{Val}^{\mathfrak{M}}(\bar{n}')$. By the definition of the value function, this is $= \iota^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n}))$. Since $\text{Val}^{\mathfrak{M}}(\bar{n}) = g(n)$ we get $g(\iota^{\mathfrak{N}}(n)) = \iota^{\mathfrak{M}}(g(n))$.
3. $g(+^{\mathfrak{N}}(n, m)) = g(n+m)$, since $+$ in \mathfrak{N} is the addition function on \mathbb{N} . Then, $g(n+m) = \text{Val}^{\mathfrak{M}}(\overline{n+m})$ by definition of g . But $\mathbf{Q} \vdash \overline{n+m} = (\bar{n} + \bar{m})$, so $\text{Val}^{\mathfrak{M}}(\overline{n+m}) = \text{Val}^{\mathfrak{M}}(\bar{n} + \bar{m})$. By the definition of the value function, this is $= +^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n}), \text{Val}^{\mathfrak{M}}(\bar{m}))$. Since $\text{Val}^{\mathfrak{M}}(\bar{n}) = g(n)$ and $\text{Val}^{\mathfrak{M}}(\bar{m}) = g(m)$, we get $g(+^{\mathfrak{N}}(n, m)) = +^{\mathfrak{M}}(g(n), g(m))$.
4. $g(\times^{\mathfrak{N}}(n, m)) = \times^{\mathfrak{M}}(g(n), g(m))$: Exercise.
5. $\langle n, m \rangle \in <^{\mathfrak{N}}$ iff $n < m$. If $n < m$, then $\mathbf{Q} \vdash \bar{n} < \bar{m}$, and also $\mathfrak{M} \models \bar{n} < \bar{m}$. Thus $\langle \text{Val}^{\mathfrak{M}}(\bar{n}), \text{Val}^{\mathfrak{M}}(\bar{m}) \rangle \in <^{\mathfrak{M}}$, i.e., $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$. If $n \not< m$, then $\mathbf{Q} \vdash \neg \bar{n} < \bar{m}$, and consequently $\mathfrak{M} \not\models \bar{n} < \bar{m}$. Thus, as before, $\langle g(n), g(m) \rangle \notin <^{\mathfrak{M}}$. Together, we get: $\langle n, m \rangle \in <^{\mathfrak{N}}$ iff $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$.

□

The function g is the most obvious way of defining a mapping from \mathbb{N} to the domain of any other **structure** \mathfrak{M} for \mathcal{L}_A , since every such \mathfrak{M} contains **elements** named by $\bar{0}$, $\bar{1}$, $\bar{2}$, etc. So it isn't surprising that if \mathfrak{M} makes at least some basic statements about the \bar{n} 's true in the same way that \mathfrak{N} does, and g is also bijective, then g will turn into an isomorphism. In fact, if $|\mathfrak{M}|$ contains no **elements** other than what the \bar{n} 's name, it's the only one.

[explanation](#)

[mod:mar:stm:](#)
[prop:thq-unique-iso](#)

Proposition mar.4. *If \mathfrak{M} is standard, then g from the proof of [Proposition mar.3](#) is the only isomorphism from \mathfrak{N} to \mathfrak{M} .*

Proof. Suppose $h: \mathbb{N} \rightarrow |\mathfrak{M}|$ is an isomorphism between \mathfrak{N} and \mathfrak{M} . We show that $g = h$ by induction on n . If $n = 0$, then $g(0) = o^{\mathfrak{M}}$ by definition of g . But since h is an isomorphism, $h(0) = h(o^{\mathfrak{N}}) = o^{\mathfrak{M}}$, so $g(0) = h(0)$.

Now consider the case for $n + 1$. We have

$$\begin{aligned}
g(n + 1) &= \text{Val}^{\mathfrak{M}}(\overline{n + 1}) \text{ by definition of } g \\
&= \text{Val}^{\mathfrak{M}}(\overline{n'}) \\
&= \iota^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\overline{n})) \\
&= \iota^{\mathfrak{M}}(g(n)) \text{ by definition of } g \\
&= \iota^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis} \\
&= h(\iota^{\mathfrak{M}}(n)) \text{ since } h \text{ is an isomorphism} \\
&= h(n + 1)
\end{aligned}$$

□

explanation For any **denumerable** set X , there's a **bijection** between \mathbb{N} and X , so every such set X is potentially the **domain** of a standard model. In fact, once you pick an object $z \in X$ and a suitable function $s: X \rightarrow X$ as $o^{\mathfrak{X}}$ and $\iota^{\mathfrak{X}}$, the interpretation of $+$, \times , and $<$ is already fixed. Only functions $s = \iota^{\mathfrak{X}}$ that are both **injective** and **surjective** are suitable in a standard model. It has to be **injective** since the successor function in \mathfrak{N} is, and that ι is **injective** is expressed by a **sentence** true in \mathfrak{N} which \mathfrak{X} thus also has to make true. It has to be **surjective** because otherwise there would be some $x \in X$ not in the domain of s , i.e., the **sentence** $\forall x \exists y y' = x$ would be false—but it is true in \mathfrak{N} .

mar.3 Non-Standard Models

explanation We call a **structure** for \mathcal{L}_A standard if it is isomorphic to \mathfrak{N} . If a **structure** isn't isomorphic to \mathfrak{N} , it is called non-standard.

Definition mar.5. A **structure** \mathfrak{M} for \mathcal{L}_A is *non-standard* if it is not isomorphic to \mathfrak{N} . The **elements** $x \in |\mathfrak{M}|$ which are equal to $\text{Val}^{\mathfrak{M}}(\overline{n})$ for some $n \in \mathbb{N}$ are called *standard numbers* (of \mathfrak{M}), and those not, *non-standard numbers*.

explanation By **Proposition mar.2**, any standard **structure** for \mathcal{L}_A contains only standard **elements**. Consequently, a non-standard **structure** must contain at least one non-standard element. In fact, the existence of a non-standard **element** guarantees that the **structure** is non-standard.

Proposition mar.6. *If a **structure** \mathfrak{M} for \mathcal{L}_A contains a non-standard number, \mathfrak{M} is non-standard.*

Proof. Suppose not, i.e., suppose \mathfrak{M} standard but contains a non-standard number x . Let $g: \mathbb{N} \rightarrow |\mathfrak{M}|$ be an isomorphism. It is easy to see (by induction on n) that $g(\text{Val}^{\mathfrak{N}}(\overline{n})) = \text{Val}^{\mathfrak{M}}(\overline{n})$. In other words, g maps standard numbers of \mathfrak{N} to standard numbers of \mathfrak{M} . If \mathfrak{M} contains a non-standard number, g cannot be **surjective**, contrary to hypothesis. □

Problem mar.2. Recall that \mathbf{Q} contains the axioms

$$\forall x \forall y (x' = y' \rightarrow x = y) \quad (Q_1)$$

$$\forall x 0 \neq x' \quad (Q_2)$$

$$\forall x (x \neq 0 \rightarrow \exists y x = y') \quad (Q_3)$$

Give **structures** $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$ such that

1. $\mathfrak{M}_1 \models Q_1, \mathfrak{M}_1 \models Q_2, \mathfrak{M}_1 \not\models Q_3$;
2. $\mathfrak{M}_2 \models Q_1, \mathfrak{M}_2 \not\models Q_2, \mathfrak{M}_2 \models Q_3$; and
3. $\mathfrak{M}_3 \not\models Q_1, \mathfrak{M}_3 \models Q_2, \mathfrak{M}_3 \models Q_3$;

Obviously, you just have to specify $0^{\mathfrak{M}_i}$ and $'^{\mathfrak{M}_i}$ for each.

It is easy enough to specify non-standard **structures** for \mathcal{L}_A . For instance, [explanation](#) take the structure with **domain** \mathbb{Z} and interpret all non-logical symbols as usual. Since negative numbers are not values of \bar{n} for any n , this structure is non-standard. Of course, it will not be a *model* of arithmetic in the sense that it makes the same sentences true as \mathfrak{N} . For instance, $\forall x x' \neq 0$ is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

Proposition mar.7. Let $\mathbf{TA} = \{\varphi : \mathfrak{N} \models \varphi\}$ be the theory of \mathbf{N} . \mathbf{TA} has *an enumerable non-standard model*.

Proof. Expand \mathcal{L}_A by a new **constant symbol** c and consider the set of **sentences**

$$\Gamma = \mathbf{TA} \cup \{c \neq \bar{0}, c \neq \bar{1}, c \neq \bar{2}, \dots\}$$

Any model \mathfrak{M}^c of Γ would contain **an element** $x = c^{\mathfrak{M}}$ which is non-standard, since $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$ for all $n \in \mathbb{N}$. Also, obviously, $\mathfrak{M}^c \models \mathbf{TA}$, since $\mathbf{TA} \subseteq \Gamma$. If we turn \mathfrak{M}^c into a **structure** \mathfrak{M} for \mathcal{L}_A simply by forgetting about c , its domain still contains the non-standard x , and also $\mathfrak{M} \models \mathbf{TA}$. The latter is guaranteed since c does not occur in \mathbf{TA} . So, it suffices to show that Γ has a model.

We use the compactness theorem to show that Γ has a model. If every finite subset of Γ is satisfiable, so is Γ . Consider any finite subset $\Gamma_0 \subseteq \Gamma$. Γ_0 includes some **sentences** of \mathbf{TA} and some of the form $c \neq \bar{k}$, but only finitely many. Suppose k is the largest number so that $c \neq \bar{k} \in \Gamma_0$. Define \mathfrak{N}_k by expanding \mathfrak{N} to include the interpretation $c^{\mathfrak{N}_k} = k + 1$. $\mathfrak{N}_k \models \Gamma_0$: if $\varphi \in \mathbf{TA}$, $\mathfrak{N}_k \models \varphi$ since \mathfrak{N}_k is just like \mathfrak{N} in all respects except c , and c does not occur in φ . And $\mathfrak{N}_k \models c \neq \bar{n}$, since $n \leq k$, and $\text{Val}^{\mathfrak{N}_k}(c) = k + 1$. Thus, every finite subset of Γ is satisfiable. \square

mar.4 Models of \mathbf{Q}

explanation

We know that there are non-standard **structures** that make the same **sentences** true as \mathfrak{N} does, i.e., is a model of **TA**. Since $\mathfrak{N} \models \mathbf{Q}$, any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g., $\mathbf{Q} \not\models \forall x \forall y (x+y) = (y+x)$. Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make $\forall x \forall y (x+y) = (y+x)$ false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly.

Example mar.8. Consider the **structure** \mathfrak{K} with domain $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

mod:mar:mdq:
ex:model-K-of-Q

$$\begin{aligned} 0^{\mathfrak{K}} &= 0 \\ \iota^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x+y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

To show that $\mathfrak{K} \models \mathbf{Q}$ we have to verify that all axioms of **Q** are true in \mathfrak{K} . For convenience, let's write x^* for $\iota^{\mathfrak{K}}(x)$ (the “successor” of x in \mathfrak{K}), $x \oplus y$ for $+^{\mathfrak{K}}(x, y)$ (the “sum” of x and y in \mathfrak{K}), $x \otimes y$ for $\times^{\mathfrak{K}}(x, y)$ (the “product” of x and y in \mathfrak{K}), and $x \oslash y$ for $\langle x, y \rangle \in <^{\mathfrak{K}}$. With these abbreviations, we can give the operations in \mathfrak{K} more perspicuously as

x	x^*	$x \oplus y$	m	a	$x \otimes y$	m	a
n	$n+1$	n	$n+m$	a	n	nm	a
a	a	a	a	a	a	a	a

We have $n \oslash m$ iff $n < m$ for $n, m \in \mathbb{N}$ and $x \oslash a$ for all $x \in |\mathfrak{K}|$.

$\mathfrak{K} \models \forall x \forall y (x' = y' \rightarrow x = y)$ since $*$ is **injective**. $\mathfrak{K} \models \forall x 0 \neq x'$ since 0 is not a $*$ -successor in \mathfrak{K} . $\mathfrak{N} \models \forall x (x \neq 0 \rightarrow \exists y x = y')$ since for every $n > 0$, $n = (n-1)^*$, and $a = a^*$.

$\mathfrak{K} \models \forall x (x \oplus 0) = x$ since $n \oplus 0 = n + 0 = n$, and $a \oplus 0 = a$ by definition of \oplus . $\mathfrak{K} \models \forall x \forall y (x \oplus y)' = (x + y)'$ is a bit trickier. If n, m are both standard, we have:

$$(n \oplus m)^* = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*$$

since \oplus and $*$ agree with $+$ and ι on standard numbers. Now suppose $x \in |\mathfrak{K}|$. Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if $y \in |\mathfrak{K}|$ but $x = a$. Here we also have to distinguish cases according to whether $y = n$ is standard or $y = b$:

$$\begin{aligned}(a \oplus n^*) &= (a \oplus (n + 1)) = a = a^* = (x \oplus n)^* \\ (a \oplus a^*) &= (a \oplus a) = a = a^* = (x \oplus a)^*\end{aligned}$$

This is of course a bit more detailed than needed. For instance, since $a \oplus z = a$ whatever z is, we can immediately conclude $a \oplus a^* = a$. The remaining axioms can be verified the same way.

\mathfrak{K} is thus a model of **Q**. Its “addition” \oplus is also commutative. But there are other sentences true in \mathfrak{N} but false in \mathfrak{K} , and vice versa. For instance, $a \otimes a$, so $\mathfrak{K} \models \exists x x < x$ and $\mathfrak{K} \not\models \forall x \neg x < x$. This shows that **Q** $\not\models \forall x \neg x < x$.

Problem mar.3. Prove that \mathfrak{K} from [Example mar.8](#) satisfies the remaining axioms of **Q**,

$$\forall x (x \times 0) = 0 \tag{Q6}$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \tag{Q7}$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (x + z' = y)) \tag{Q8}$$

Find a sentence only involving \prime true in \mathfrak{N} but false in \mathfrak{K} .

[mod:mar:mdq:](#)
[ex:model-L-of-Q](#)

Example mar.9. Consider the **structure** \mathfrak{L} with domain $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$ and interpretations $\prime^{\mathfrak{L}} = *$, $+^{\mathfrak{L}} = \oplus$ given by

x	x^*	$x \oplus y$	m	a	b
n	$n + 1$	n	$n + m$	b	a
a	a	a	a	b	a
b	b	b	b	b	a

Since $*$ is **injective**, 0 is not in its range, and every $x \in |\mathfrak{L}|$ other than 0 is, axioms Q_1 – Q_3 are true in \mathfrak{L} . For any x , $x \oplus 0 = x$, so Q_4 is true as well. For Q_5 , consider $x \oplus y^*$ and $(x \oplus y)^*$. They are equal if x and y are both standard, since then $*$ and \oplus agree with \prime and $+$. If x is non-standard, and y is standard, we have $x \oplus y^* = x = x^* = (x \oplus y)^*$. If x and y are both non-standard, we have four cases:

$$\begin{aligned}a \oplus a^* &= b = b^* = (a \oplus a)^* \\ b \oplus b^* &= a = a^* = (b \oplus b)^* \\ b \oplus a^* &= b = b^* = (b \oplus y)^* \\ a \oplus b^* &= a = a^* = (a \oplus b)^*\end{aligned}$$

If x is standard, but y is non-standard, we have

$$\begin{aligned}n \oplus a^* &= n \oplus a = b = b^* = (n \oplus a)^* \\ n \oplus b^* &= n \oplus b = a = a^* = (n \oplus b)^*\end{aligned}$$

So, $\mathfrak{L} \models Q_5$. However, $a \oplus 0 \neq 0 \oplus a$, so $\mathfrak{L} \not\models \forall x \forall y (x + y) = (y + x)$.

Problem mar.4. Expand \mathcal{L} of [Example mar.9](#) to include \otimes and \ominus that interpret \times and $<$. Show that your structure satisfies the remaining axioms of \mathbf{Q} ,

$$\forall x (x \times 0) = 0 \tag{Q6}$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \tag{Q7}$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (x + z' = y)) \tag{Q8}$$

Problem mar.5. In \mathcal{L} of [Example mar.9](#), $a^* = a$ and $b^* = b$. Is there a model of \mathbf{Q} in which $a^* = b$ and $b^* = a$?

explanation We've explicitly constructed models of \mathbf{Q} in which the non-standard **elements** live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard **element** x cannot be $\ominus 0$. Otherwise, for some z , $x \oplus z^* = 0$ by $Q8$. But then $0 = x \oplus z^* = (x \oplus z)^*$ by $Q5$, contradicting $Q2$. Also, for every n , $\mathbf{Q} \vdash \forall x (x < \bar{n}' \rightarrow (x = \bar{0} \vee x = \bar{1} \vee \dots \vee x = \bar{n}))$, so we can't have $a \ominus n$ for any $n > 0$.

mar.5 Computable Models of Arithmetic

explanation The standard model \mathfrak{N} has two nice features. Its domain is the natural numbers \mathbb{N} , i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral \bar{n} actually picks out n . The other nice feature is that the interpretations of the non-logical symbols of \mathcal{L}_A are all *computable*. The successor, addition, and multiplication functions which serve as $s^{\mathfrak{N}}$, $+$ ^{\mathfrak{N}} , and $\times^{\mathfrak{N}}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on \mathfrak{N} , i.e., $<^{\mathfrak{N}}$, is decidable.

Non-standard models of arithmetical theories such as \mathbf{Q} and \mathbf{PA} must contain non-standard elements. Thus their domains typically include **elements** in addition to \mathbb{N} . However, any countable **structure** can be built on any **denumerable** set, including \mathbb{N} . So there are also non-standard models with domain \mathbb{N} . In such models \mathfrak{M} , of course, at least some numbers cannot play the roles they usually play, since some k must be different from $\text{Val}^{\mathfrak{M}}(\bar{n})$ for all $n \in \mathbb{N}$.

Definition mar.10. A **structure** \mathfrak{M} for \mathcal{L}_A is *computable* iff $|\mathfrak{M}| = \mathbb{N}$ and $s^{\mathfrak{M}}$, $+$ ^{\mathfrak{M}} , $\times^{\mathfrak{M}}$ are computable functions and $<^{\mathfrak{M}}$ is a decidable relation.

Example mar.11. Recall the structure \mathfrak{K} from [Example mar.8](#) Its domain

was $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$$\begin{aligned} o^{\mathfrak{K}} &= 0 \\ \iota^{\mathfrak{K}}(x) &= \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

But $|\mathfrak{K}|$ is **denumerable** and so is equinumerous with \mathbb{N} . For instance, $g: \mathbb{N} \rightarrow |\mathfrak{K}|$ with $g(0) = a$ and $g(n) = n + 1$ for $n > 0$ is a **bijection**. We can turn it into an isomorphism between a new model \mathfrak{K}' of \mathbf{Q} and \mathfrak{K} . In \mathfrak{K}' , we have to assign different functions and relations to the symbols of \mathcal{L}_A , since different **elements** of \mathbb{N} play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of a , not of the smallest standard number. The smallest standard number is now 1. So we assign $o^{\mathfrak{K}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n + 1$. But 0 now plays the role of a , which is its own successor. So $\iota^{\mathfrak{K}'}(0) = 0$. For addition and multiplication we likewise have

$$\begin{aligned} +^{\mathfrak{K}'}(x, y) &= \begin{cases} x + y & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}'}(x, y) &= \begin{cases} xy & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

And we have $\langle x, y \rangle \in <^{\mathfrak{K}'}$ iff $x < y$ and $x > 0$ and $y > 0$, or if $y = 0$.

All of these functions are computable functions of natural numbers and $<^{\mathfrak{K}'}$ is a decidable relation on \mathbb{N} —but they are not the same functions as successor, addition, and multiplication on \mathbb{N} , and $<^{\mathfrak{K}'}$ is not the same relation as $<$ on \mathbb{N} .

Problem mar.6. Give a **structure** \mathfrak{L}' with $|\mathfrak{L}'| = \mathbb{N}$ isomorphic to \mathfrak{L} of [Example mar.9](#).

This example shows that \mathbf{Q} has computable non-standard models with domain \mathbb{N} . However, the following result shows that this is not true for models of \mathbf{PA} (and thus also for models of \mathbf{TA}). explanation

Theorem mar.12 (Tennenbaum's Theorem). *\mathfrak{N} is the only computable model of \mathbf{PA} .*

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Bibliography