

## Chapter udf

# Models of Arithmetic

### mar.1 Introduction

The *standard model* of arithmetic is the **structure**  $\mathfrak{N}$  with  $|\mathfrak{N}| = \mathbb{N}$  in which  $o$ ,  $l$ ,  $+$ ,  $\times$ , and  $<$  are interpreted as you would expect. That is,  $o$  is 0,  $l$  is the successor function,  $+$  is interpreted as addition and  $\times$  as multiplication of the numbers in  $\mathbb{N}$ . Specifically,

$$\begin{aligned}o^{\mathfrak{N}} &= 0 \\l^{\mathfrak{N}}(n) &= n + 1 \\+^{\mathfrak{N}}(n, m) &= n + m \\\times^{\mathfrak{N}}(n, m) &= nm\end{aligned}$$

Of course, there are structures for  $\mathcal{L}_A$  that have domains other than  $\mathbb{N}$ . For instance, we can take  $\mathfrak{M}$  with domain  $|\mathfrak{M}| = \{a\}^*$  (the finite sequences of the single symbol  $a$ , i.e.,  $\emptyset, a, aa, aaa, \dots$ ), and interpretations

$$\begin{aligned}o^{\mathfrak{M}} &= \emptyset \\l^{\mathfrak{M}}(s) &= s \frown a \\+^{\mathfrak{M}}(n, m) &= a^{n+m} \\\times^{\mathfrak{M}}(n, m) &= a^{nm}\end{aligned}$$

These two structures are “essentially the same” in the sense that the only difference is the **elements** of the **domains** but not how the **elements** of the **domains** are related among each other by the interpretation functions. We say that the two **structures** are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in  $\mathfrak{N}$  also has models that are not isomorphic to  $\mathfrak{N}$ . Such structures are called *non-standard*. The interesting thing about them is that while the **elements** of a standard model (i.e.,  $\mathfrak{N}$ , but also all **structures** isomorphic to it) are exhausted by the values of the standard numerals  $\bar{n}$ , i.e.,

$$|\mathfrak{N}| = \{\text{Val}^{\mathfrak{N}}(\bar{n}) : n \in \mathbb{N}\}$$

that isn't the case in non-standard models: if  $\mathfrak{M}$  is non-standard, then there is at least one  $x \in |\mathfrak{M}|$  such that  $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$  for all  $n$ .

These non-standard elements are pretty neat: they are “infinite natural numbers.” But their existence also explains, in a sense, the incompleteness phenomena. Consider an example, e.g., the consistency statement for Peano arithmetic,  $\text{Con}_{\mathbf{PA}}$ , i.e.,  $\neg \exists x \text{Prf}_{\mathbf{PA}}(x, \ulcorner \perp \urcorner)$ . Since  $\mathbf{PA}$  neither proves  $\text{Con}_{\mathbf{PA}}$  nor  $\neg \text{Con}_{\mathbf{PA}}$ , either can be consistently added to  $\mathbf{PA}$ . Since  $\mathbf{PA}$  is consistent,  $\mathfrak{N} \models \text{Con}_{\mathbf{PA}}$ , and consequently  $\mathfrak{N} \not\models \neg \text{Con}_{\mathbf{PA}}$ . So  $\mathfrak{N}$  is *not* a model of  $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$ , and all its models must be nonstandard. Models of  $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$  must contain some **element** that serves as the witness that makes  $\exists x \text{Prf}_{\mathbf{PA}}(\ulcorner \perp \urcorner)$  true, i.e., a Gödel number of a **derivation** of a contradiction from  $\mathbf{PA}$ . Such an **element** can't be standard—since  $\mathbf{PA} \vdash \neg \text{Prf}_{\mathbf{PA}}(\bar{n}, \ulcorner \perp \urcorner)$  for every  $n$ .

## mar.2 Standard Models of Arithmetic

The language of arithmetic  $\mathcal{L}_A$  is obviously intended to be about numbers, specifically, about natural numbers. So, “the” standard model  $\mathfrak{N}$  is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two **structures** that are isomorphic share those. So we can be a bit more liberal, and consider any **structure** that is isomorphic to  $\mathfrak{N}$  “standard.”

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sec

**Definition mar.1.** A **structure** for  $\mathcal{L}_A$  is *standard* if it is isomorphic to  $\mathfrak{N}$ .

**Proposition mar.2.** If a **structure**  $\mathfrak{M}$  is standard, its domain is the set of values of the standard numerals, i.e.,

mod:mar:stm:  
prop:standard-domain

$$|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$$

*Proof.* Clearly, every  $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$ . We just have to show that every  $x \in |\mathfrak{M}|$  is equal to  $\text{Val}^{\mathfrak{M}}(\bar{n})$  for some  $n$ . Since  $\mathfrak{M}$  is standard, it is isomorphic to  $\mathfrak{N}$ . Suppose  $g: \mathbb{N} \rightarrow |\mathfrak{M}|$  is an isomorphism. Then  $g(n) = g(\text{Val}^{\mathfrak{N}}(\bar{n})) = \text{Val}^{\mathfrak{M}}(\bar{n})$ . But for every  $x \in |\mathfrak{M}|$ , there is an  $n \in \mathbb{N}$  such that  $g(n) = x$ , since  $g$  is **surjective**.  $\square$

explanation

If a structure  $\mathfrak{M}$  for  $\mathcal{L}_A$  is standard, the elements of its **domain** can all be named by the standard numerals  $\bar{0}, \bar{1}, \bar{2}, \dots$ , i.e., the terms  $o, o', o'', \dots$ . Of course, this does not mean that the **elements** of  $|\mathfrak{M}|$  are the numbers, just that we can pick them out the same way we can pick out the numbers in  $|\mathfrak{N}|$ .

**Problem mar.1.** Show that the converse of **Proposition mar.2** is false, i.e., give an example of a **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$  that is not isomorphic to  $\mathfrak{N}$ .

**Proposition mar.3.** If  $\mathfrak{M} \models \mathbf{Q}$ , and  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ , then  $\mathfrak{M}$  is *standard*.

mod:mar:stm:  
prop:thq-standard

*Proof.* We have to show that  $\mathfrak{M}$  is isomorphic to  $\mathfrak{N}$ . Consider the function  $g: \mathbb{N} \rightarrow |\mathfrak{M}|$  defined by  $g(n) = \text{Val}^{\mathfrak{M}}(\bar{n})$ . By the hypothesis,  $g$  is **surjective**. It is also **injective**:  $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$  whenever  $n \neq m$ . Thus, since  $\mathfrak{M} \models \mathbf{Q}$ ,  $\mathfrak{M} \models \bar{n} \neq \bar{m}$ , whenever  $n \neq m$ . Thus, if  $n \neq m$ , then  $\text{Val}^{\mathfrak{M}}(\bar{n}) \neq \text{Val}^{\mathfrak{M}}(\bar{m})$ , i.e.,  $g(n) \neq g(m)$ .

We also have to verify that  $g$  is an isomorphism.

1. We have  $g(o^{\mathfrak{N}}) = g(0)$  since,  $o^{\mathfrak{N}} = 0$ . By definition of  $g$ ,  $g(0) = \text{Val}^{\mathfrak{M}}(\bar{0})$ . But  $\bar{0}$  is just  $o$ , and the value of a term which happens to be a **constant symbol** is given by what the **structure** assigns to that **constant symbol**, i.e.,  $\text{Val}^{\mathfrak{M}}(o) = o^{\mathfrak{M}}$ . So we have  $g(o^{\mathfrak{N}}) = o^{\mathfrak{M}}$  as required.
2.  $g(\iota^{\mathfrak{N}}(n)) = g(n+1)$ , since  $\iota$  in  $\mathfrak{N}$  is the successor function on  $\mathbb{N}$ . Then,  $g(n+1) = \text{Val}^{\mathfrak{M}}(\overline{n+1})$  by definition of  $g$ . But  $\overline{n+1}$  is the same term as  $\bar{n}'$ , so  $\text{Val}^{\mathfrak{M}}(\overline{n+1}) = \text{Val}^{\mathfrak{M}}(\bar{n}')$ . By the definition of the value function, this is  $= \iota^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n}))$ . Since  $\text{Val}^{\mathfrak{M}}(\bar{n}) = g(n)$  we get  $g(\iota^{\mathfrak{N}}(n)) = \iota^{\mathfrak{M}}(g(n))$ .
3.  $g(+^{\mathfrak{N}}(n, m)) = g(n+m)$ , since  $+$  in  $\mathfrak{N}$  is the addition function on  $\mathbb{N}$ . Then,  $g(n+m) = \text{Val}^{\mathfrak{M}}(\overline{n+m})$  by definition of  $g$ . But  $\mathbf{Q} \vdash \bar{n} + \bar{m} = (\bar{n} + \bar{m})$ , so  $\text{Val}^{\mathfrak{M}}(\overline{n+m}) = \text{Val}^{\mathfrak{M}}(\bar{n} + \bar{m})$ . By the definition of the value function, this is  $= +^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\bar{n}), \text{Val}^{\mathfrak{M}}(\bar{m}))$ . Since  $\text{Val}^{\mathfrak{M}}(\bar{n}) = g(n)$  and  $\text{Val}^{\mathfrak{M}}(\bar{m}) = g(m)$ , we get  $g(+^{\mathfrak{N}}(n, m)) = +^{\mathfrak{M}}(g(n), g(m))$ .
4.  $g(\times^{\mathfrak{N}}(n, m)) = \times^{\mathfrak{M}}(g(n), g(m))$ : Exercise.
5.  $\langle n, m \rangle \in <^{\mathfrak{N}}$  iff  $n < m$ . If  $n < m$ , then  $\mathbf{Q} \vdash \bar{n} < \bar{m}$ , and also  $\mathfrak{M} \models \bar{n} < \bar{m}$ . Thus  $\langle \text{Val}^{\mathfrak{M}}(\bar{n}), \text{Val}^{\mathfrak{M}}(\bar{m}) \rangle \in <^{\mathfrak{M}}$ , i.e.,  $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$ . If  $n \not< m$ , then  $\mathbf{Q} \vdash \neg \bar{n} < \bar{m}$ , and consequently  $\mathfrak{M} \not\models \bar{n} < \bar{m}$ . Thus, as before,  $\langle g(n), g(m) \rangle \notin <^{\mathfrak{M}}$ . Together, we get:  $\langle n, m \rangle \in <^{\mathfrak{N}}$  iff  $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$ .

□

The function  $g$  is the most obvious way of defining a mapping from  $\mathbb{N}$  to the domain of any other **structure**  $\mathfrak{M}$  for  $\mathcal{L}_A$ , since every such  $\mathfrak{M}$  contains **elements** named by  $\bar{0}$ ,  $\bar{1}$ ,  $\bar{2}$ , etc. So it isn't surprising that if  $\mathfrak{M}$  makes at least some basic statements about the  $\bar{n}$ 's true in the same way that  $\mathfrak{N}$  does, and  $g$  is also bijective, then  $g$  will turn into an isomorphism. In fact, if  $|\mathfrak{M}|$  contains no **elements** other than what the  $\bar{n}$ 's name, it's the only one. explanation

mod:mar:stm: prop:thq-unique-iso **Proposition mar.4.** *If  $\mathfrak{M}$  is standard, then  $g$  from the proof of Proposition mar.3 is the only isomorphism from  $\mathfrak{N}$  to  $\mathfrak{M}$ .*

*Proof.* Suppose  $h: \mathbb{N} \rightarrow |\mathfrak{M}|$  is an isomorphism between  $\mathfrak{N}$  and  $\mathfrak{M}$ . We show that  $g = h$  by induction on  $n$ . If  $n = 0$ , then  $g(0) = o^{\mathfrak{M}}$  by definition of  $g$ . But since  $h$  is an isomorphism,  $h(0) = h(o^{\mathfrak{N}}) = o^{\mathfrak{M}}$ , so  $g(0) = h(0)$ .

Now consider the case for  $n + 1$ . We have

$$\begin{aligned}
g(n + 1) &= \text{Val}^{\mathfrak{M}}(\overline{n + 1}) \text{ by definition of } g \\
&= \text{Val}^{\mathfrak{M}}(\overline{n'}) \\
&= \iota^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\overline{n})) \\
&= \iota^{\mathfrak{M}}(g(n)) \text{ by definition of } g \\
&= \iota^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis} \\
&= h(\iota^{\mathfrak{N}}(n)) \text{ since } h \text{ is an isomorphism} \\
&= h(n + 1)
\end{aligned}$$

□

**explanation** For any denumerable set  $X$ , there's a bijection between  $\mathbb{N}$  and  $X$ , so every such set  $X$  is potentially the domain of a standard model. In fact, once you pick an object  $z \in X$  and a suitable function  $s: X \rightarrow X$  as  $o^{\mathfrak{X}}$  and  $\iota^{\mathfrak{X}}$ , the interpretation of  $+$ ,  $\times$ , and  $<$  is already fixed. Only functions  $s = \iota^{\mathfrak{X}}$  that are both injective and surjective are suitable in a standard model. It has to be injective since the successor function in  $\mathfrak{N}$  is, and that  $\iota$  is injective is expressed by a sentence true in  $\mathfrak{N}$  which  $\mathfrak{X}$  thus also has to make true. It has to be surjective because otherwise there would be some  $x \in X$  not in the domain of  $s$ , i.e., the sentence  $\forall x \exists y y' = x$  would be false—but it is true in  $\mathfrak{N}$ .

### mar.3 Non-Standard Models

**explanation** We call a structure for  $\mathcal{L}_A$  standard if it is isomorphic to  $\mathfrak{N}$ . If a structure isn't isomorphic to  $\mathfrak{N}$ , it is called non-standard.

**Definition mar.5.** A structure  $\mathfrak{M}$  for  $\mathcal{L}_A$  is *non-standard* if it is not isomorphic to  $\mathfrak{N}$ . The elements  $x \in |\mathfrak{M}|$  which are equal to  $\text{Val}^{\mathfrak{M}}(\overline{n})$  for some  $n \in \mathbb{N}$  are called *standard numbers* (of  $\mathfrak{M}$ ), and those not, *non-standard numbers*.

**explanation** By Proposition mar.2, any standard structure for  $\mathcal{L}_A$  contains only standard elements. Consequently, a non-standard structure must contain at least one non-standard element. In fact, the existence of a non-standard element guarantees that the structure is non-standard.

**Proposition mar.6.** *If a structure  $\mathfrak{M}$  for  $\mathcal{L}_A$  contains a non-standard number,  $\mathfrak{M}$  is non-standard.*

*Proof.* Suppose not, i.e., suppose  $\mathfrak{M}$  standard but contains a non-standard number  $x$ . Let  $g: \mathbb{N} \rightarrow |\mathfrak{M}|$  be an isomorphism. It is easy to see (by induction on  $n$ ) that  $g(\text{Val}^{\mathfrak{N}}(\overline{n})) = \text{Val}^{\mathfrak{M}}(\overline{n})$ . In other words,  $g$  maps standard numbers of  $\mathfrak{N}$  to standard numbers of  $\mathfrak{M}$ . If  $\mathfrak{M}$  contains a non-standard number,  $g$  cannot be surjective, contrary to hypothesis. □

**Problem mar.2.** Recall that  $\mathbf{Q}$  contains the axioms

$$\forall x \forall y (x' = y' \rightarrow x = y) \quad (Q_1)$$

$$\forall x 0 \neq x' \quad (Q_2)$$

$$\forall x (x \neq 0 \rightarrow \exists y x = y') \quad (Q_3)$$

Give structures  $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$  such that

1.  $\mathfrak{M}_1 \models Q_1, \mathfrak{M}_1 \models Q_2, \mathfrak{M}_1 \not\models Q_3$ ;
2.  $\mathfrak{M}_2 \models Q_1, \mathfrak{M}_2 \not\models Q_2, \mathfrak{M}_2 \models Q_3$ ; and
3.  $\mathfrak{M}_3 \not\models Q_1, \mathfrak{M}_3 \models Q_2, \mathfrak{M}_3 \models Q_3$ ;

Obviously, you just have to specify  $0^{\mathfrak{M}_i}$  and  $'^{\mathfrak{M}_i}$  for each.

It is easy enough to specify non-standard structures for  $\mathcal{L}_A$ . For instance, [explanation](#) take the structure with domain  $\mathbb{Z}$  and interpret all non-logical symbols as usual. Since negative numbers are not values of  $\bar{n}$  for any  $n$ , this structure is non-standard. Of course, it will not be a *model* of arithmetic in the sense that it makes the same sentences true as  $\mathfrak{N}$ . For instance,  $\forall x x' \neq 0$  is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

**Proposition mar.7.** Let  $\mathbf{TA} = \{\varphi : \mathfrak{N} \models \varphi\}$  be the theory of  $\mathbb{N}$ .  $\mathbf{TA}$  has an *enumerable non-standard model*.

*Proof.* Expand  $\mathcal{L}_A$  by a new constant symbol  $c$  and consider the set of sentences

$$\Gamma = \mathbf{TA} \cup \{c \neq \bar{0}, c \neq \bar{1}, c \neq \bar{2}, \dots\}$$

Any model  $\mathfrak{M}^c$  of  $\Gamma$  would contain an element  $x = c^{\mathfrak{M}}$  which is non-standard, since  $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$  for all  $n \in \mathbb{N}$ . Also, obviously,  $\mathfrak{M}^c \models \mathbf{TA}$ , since  $\mathbf{TA} \subseteq \Gamma$ . If we turn  $\mathfrak{M}^c$  into a structure  $\mathfrak{M}$  for  $\mathcal{L}_A$  simply by forgetting about  $c$ , its domain still contains the non-standard  $x$ , and also  $\mathfrak{M} \models \mathbf{TA}$ . The latter is guaranteed since  $c$  does not occur in  $\mathbf{TA}$ . So, it suffices to show that  $\Gamma$  has a model.

We use the compactness theorem to show that  $\Gamma$  has a model. If every finite subset of  $\Gamma$  is satisfiable, so is  $\Gamma$ . Consider any finite subset  $\Gamma_0 \subseteq \Gamma$ .  $\Gamma_0$  includes some sentences of  $\mathbf{TA}$  and some of the form  $c \neq \bar{k} \in \Gamma_0$ . Define  $\mathfrak{N}_k$  by expanding  $\mathfrak{N}$  to include the interpretation  $c^{\mathfrak{N}_k} = k + 1$ .  $\mathfrak{N}_k \models \Gamma_0$ : if  $\varphi \in \mathbf{TA}$ ,  $\mathfrak{N}_k \models \varphi$  since  $\mathfrak{N}_k$  is just like  $\mathfrak{N}$  in all respects except  $c$ , and  $c$  does not occur in  $\varphi$ . And  $\mathfrak{N}_k \models c \neq \bar{n}$ , since  $n \leq k$ , and  $\text{Val}^{\mathfrak{N}_k}(c) = k + 1$ . Thus, every finite subset of  $\Gamma$  is satisfiable.  $\square$

## mar.4 Models of $\mathbf{Q}$

explanation

We know that there are non-standard **structures** that make the same **sentences** true as  $\mathfrak{N}$  does, i.e., is a model of **TA**. Since  $\mathfrak{N} \models \mathbf{Q}$ , any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g.,  $\mathbf{Q} \not\models \forall x \forall y (x+y) = (y+x)$ . Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make  $\forall x \forall y (x+y) = (y+x)$  false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly.

**Example mar.8.** Consider the **structure**  $\mathfrak{K}$  with domain  $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$  and interpretations

mod:mar:mdq:  
ex:model-K-of-Q

$$\begin{aligned} 0^{\mathfrak{K}} &= 0 \\ \iota^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x+y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

To show that  $\mathfrak{K} \models \mathbf{Q}$  we have to verify that all axioms of **Q** are true in  $\mathfrak{K}$ . For convenience, let's write  $x^*$  for  $\iota^{\mathfrak{K}}(x)$  (the “successor” of  $x$  in  $\mathfrak{K}$ ),  $x \oplus y$  for  $+^{\mathfrak{K}}(x, y)$  (the “sum” of  $x$  and  $y$  in  $\mathfrak{K}$ ),  $x \otimes y$  for  $\times^{\mathfrak{K}}(x, y)$  (the “product” of  $x$  and  $y$  in  $\mathfrak{K}$ ), and  $x \oslash y$  for  $\langle x, y \rangle \in <^{\mathfrak{K}}$ . With these abbreviations, we can give the operations in  $\mathfrak{K}$  more perspicuously as

$x$	$x^*$	$x \oplus y$	$m$	$a$	$x \otimes y$	$m$	$a$
$n$	$n+1$	$n$	$n+m$	$a$	$n$	$nm$	$a$
$a$	$a$	$a$	$a$	$a$	$a$	$a$	$a$

We have  $n \oslash m$  iff  $n < m$  for  $n, m \in \mathbb{N}$  and  $x \oslash a$  for all  $x \in |\mathfrak{K}|$ .

$\mathfrak{K} \models \forall x \forall y (x' = y' \rightarrow x = y)$  since  $*$  is **injective**.  $\mathfrak{K} \models \forall x 0 \neq x'$  since 0 is not a  $*$ -successor in  $\mathfrak{K}$ .  $\mathfrak{N} \models \forall x (x \neq 0 \rightarrow \exists y x = y')$  since for every  $n > 0$ ,  $n = (n-1)^*$ , and  $a = a^*$ .

$\mathfrak{K} \models \forall x (x + 0) = x$  since  $n \oplus 0 = n + 0 = n$ , and  $a \oplus 0 = a$  by definition of  $\oplus$ .  $\mathfrak{K} \models \forall x \forall y (x + y') = (x + y)'$  is a bit trickier. If  $n, m$  are both standard, we have:

$$(n \oplus m^*) = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*$$

since  $\oplus$  and  $*$  agree with  $+$  and  $\iota$  on standard numbers. Now suppose  $x \in |\mathfrak{K}|$ . Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if  $y \in |\mathfrak{K}|$  but  $x = a$ . Here we also have to distinguish cases according to whether  $y = n$  is standard or  $y = b$ :

$$\begin{aligned}(a \oplus n^*) &= (a \oplus (n + 1)) = a = a^* = (x \oplus n)^* \\ (a \oplus a^*) &= (a \oplus a) = a = a^* = (x \oplus a)^*\end{aligned}$$

This is of course a bit more detailed than needed. For instance, since  $a \oplus z = a$  whatever  $z$  is, we can immediately conclude  $a \oplus a^* = a$ . The remaining axioms can be verified the same way.

$\mathfrak{K}$  is thus a model of **Q**. Its “addition”  $\oplus$  is also commutative. But there are other sentences true in  $\mathfrak{N}$  but false in  $\mathfrak{K}$ , and vice versa. For instance,  $a \otimes a$ , so  $\mathfrak{K} \models \exists x x < x$  and  $\mathfrak{K} \not\models \forall x \neg x < x$ . This shows that **Q**  $\not\models \forall x \neg x < x$ .

**Problem mar.3.** Prove that  $\mathfrak{K}$  from [Example mar.8](#) satisfies the remaining axioms of **Q**,

$$\forall x (x \times 0) = 0 \tag{Q6}$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \tag{Q7}$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (x + z' = y)) \tag{Q8}$$

Find a sentence only involving  $\prime$  true in  $\mathfrak{N}$  but false in  $\mathfrak{K}$ .

[mod:mar:mdq:](#)  
[ex:model-L-of-Q](#)

**Example mar.9.** Consider the **structure**  $\mathfrak{L}$  with domain  $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$  and interpretations  $\prime^{\mathfrak{L}} = *$ ,  $+^{\mathfrak{L}} = \oplus$  given by

$x$	$x^*$	$x \oplus y$	$m$	$a$	$b$
$n$	$n + 1$	$n$	$n + m$	$b$	$a$
$a$	$a$	$a$	$a$	$b$	$a$
$b$	$b$	$b$	$b$	$b$	$a$

Since  $*$  is **injective**, 0 is not in its range, and every  $x \in |\mathfrak{L}|$  other than 0 is, axioms  $Q_1$ – $Q_3$  are true in  $\mathfrak{L}$ . For any  $x$ ,  $x \oplus 0 = x$ , so  $Q_4$  is true as well. For  $Q_5$ , consider  $x \oplus y^*$  and  $(x \oplus y)^*$ . They are equal if  $x$  and  $y$  are both standard, since then  $*$  and  $\oplus$  agree with  $\prime$  and  $+$ . If  $x$  is non-standard, and  $y$  is standard, we have  $x \oplus y^* = x = x^* = (x \oplus y)^*$ . If  $x$  and  $y$  are both non-standard, we have four cases:

$$\begin{aligned}a \oplus a^* &= b = b^* = (a \oplus a)^* \\ b \oplus b^* &= a = a^* = (b \oplus b)^* \\ b \oplus a^* &= b = b^* = (b \oplus y)^* \\ a \oplus b^* &= a = a^* = (a \oplus b)^*\end{aligned}$$

If  $x$  is standard, but  $y$  is non-standard, we have

$$\begin{aligned}n \oplus a^* &= n \oplus a = b = b^* = (n \oplus a)^* \\ n \oplus b^* &= n \oplus b = a = a^* = (n \oplus b)^*\end{aligned}$$

So,  $\mathfrak{L} \models Q_5$ . However,  $a \oplus 0 \neq 0 \oplus a$ , so  $\mathfrak{L} \not\models \forall x \forall y (x + y) = (y + x)$ .

**Problem mar.4.** Expand  $\mathcal{L}$  of [Example mar.9](#) to include  $\otimes$  and  $\ominus$  that interpret  $\times$  and  $<$ . Show that your structure satisfies the remaining axioms of  $\mathbf{Q}$ ,

$$\forall x (x \times 0) = 0 \quad (Q_6)$$

$$\forall x \forall y (x \times y') = ((x \times y) + x) \quad (Q_7)$$

$$\forall x \forall y (x < y \leftrightarrow \exists z (x + z' = y)) \quad (Q_8)$$

**Problem mar.5.** In  $\mathcal{L}$  of [Example mar.9](#),  $a^* = a$  and  $b^* = b$ . Is there a model of  $\mathbf{Q}$  in which  $a^* = b$  and  $b^* = a$ ?

**explanation** We've explicitly constructed models of  $\mathbf{Q}$  in which the non-standard **elements** live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard **element**  $x$  cannot be  $\ominus 0$ . Otherwise, for some  $z$ ,  $x \oplus z^* = 0$  by  $Q_8$ . But then  $0 = x \oplus z^* = (x \oplus z)^*$  by  $Q_5$ , contradicting  $Q_2$ . Also, for every  $n$ ,  $\mathbf{Q} \vdash \forall x (x < \bar{n}' \rightarrow (x = \bar{0} \vee x = \bar{1} \vee \dots \vee x = \bar{n}))$ , so we can't have  $a \ominus n$  for any  $n > 0$ .

## mar.5 Computable Models of Arithmetic

**explanation** The standard model  $\mathfrak{N}$  has two nice features. Its domain is the natural numbers  $\mathbb{N}$ , i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral  $\bar{n}$  actually picks out  $n$ . The other nice feature is that the interpretations of the non-logical symbols of  $\mathcal{L}_A$  are all *computable*. The successor, addition, and multiplication functions which serve as  $s^{\mathfrak{N}}$ ,  $+$  <sup>$\mathfrak{N}$</sup> , and  $\times^{\mathfrak{N}}$  are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on  $\mathfrak{N}$ , i.e.,  $<^{\mathfrak{N}}$ , is decidable.

Non-standard models of arithmetical theories such as  $\mathbf{Q}$  and  $\mathbf{PA}$  must contain non-standard elements. Thus their domains typically include **elements** in addition to  $\mathbb{N}$ . However, any countable **structure** can be built on any **denumerable** set, including  $\mathbb{N}$ . So there are also non-standard models with domain  $\mathbb{N}$ . In such models  $\mathfrak{M}$ , of course, at least some numbers cannot play the roles they usually play, since some  $k$  must be different from  $\text{Val}^{\mathfrak{M}}(\bar{n})$  for all  $n \in \mathbb{N}$ .

**Definition mar.10.** A **structure**  $\mathfrak{M}$  for  $\mathcal{L}_A$  is *computable* iff  $|\mathfrak{M}| = \mathbb{N}$  and  $s^{\mathfrak{M}}$ ,  $+$  <sup>$\mathfrak{M}$</sup> ,  $\times^{\mathfrak{M}}$  are computable functions and  $<^{\mathfrak{M}}$  is a decidable relation.

**Example mar.11.** Recall the structure  $\mathfrak{K}$  from [Example mar.8](#) Its domain



was  $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$  and interpretations

$$\begin{aligned} \circ^{\mathfrak{K}} &= 0 \\ \iota^{\mathfrak{K}}(x) &= \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

But  $|\mathfrak{K}|$  is **denumerable** and so is equinumerous with  $\mathbb{N}$ . For instance,  $g: \mathbb{N} \rightarrow |\mathfrak{K}|$  with  $g(0) = a$  and  $g(n) = n + 1$  for  $n > 0$  is a **bijection**. We can turn it into an isomorphism between a new model  $\mathfrak{K}'$  of  $\mathbf{Q}$  and  $\mathfrak{K}$ . In  $\mathfrak{K}'$ , we have to assign different functions and relations to the symbols of  $\mathcal{L}_A$ , since different **elements** of  $\mathbb{N}$  play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of  $a$ , not of the smallest standard number. The smallest standard number is now 1. So we assign  $\circ^{\mathfrak{K}'} = 1$ . The successor function is also different now: given a standard number, i.e., an  $n > 0$ , it still returns  $n + 1$ . But 0 now plays the role of  $a$ , which is its own successor. So  $\iota^{\mathfrak{K}'}(0) = 0$ . For addition and multiplication we likewise have

$$\begin{aligned} +^{\mathfrak{K}'}(x, y) &= \begin{cases} x + y & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}'}(x, y) &= \begin{cases} xy & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

And we have  $\langle x, y \rangle \in <^{\mathfrak{K}'}$  iff  $x < y$  and  $x > 0$  and  $y > 0$ , or if  $y = 0$ .

All of these functions are computable functions of natural numbers and  $<^{\mathfrak{K}'}$  is a decidable relation on  $\mathbb{N}$ —but they are not the same functions as successor, addition, and multiplication on  $\mathbb{N}$ , and  $<^{\mathfrak{K}'}$  is not the same relation as  $<$  on  $\mathbb{N}$ .

**Problem mar.6.** Give a **structure**  $\mathfrak{L}'$  with  $|\mathfrak{L}'| = \mathbb{N}$  isomorphic to  $\mathfrak{L}$  of [Example mar.9](#).

This example shows that  $\mathbf{Q}$  has computable non-standard models with domain  $\mathbb{N}$ . However, the following result shows that this is not true for models of  $\mathbf{PA}$  (and thus also for models of  $\mathbf{TA}$ ). explanation

**Theorem mar.12** (Tennenbaum's Theorem).  *$\mathfrak{N}$  is the only computable model of  $\mathbf{PA}$ .*

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# Bibliography