

mar.1 Introduction

The *standard model* of arithmetic is the **structure** \mathfrak{N} with $|\mathfrak{N}| = \mathbb{N}$ in which o , l , $+$, \times , and $<$ are interpreted as you would expect. That is, o is 0, l is the successor function, $+$ is interpreted as addition and \times as multiplication of the numbers in \mathbb{N} . Specifically,

$$\begin{aligned} o^{\mathfrak{N}} &= 0 \\ l^{\mathfrak{N}}(n) &= n + 1 \\ +^{\mathfrak{N}}(n, m) &= n + m \\ \times^{\mathfrak{N}}(n, m) &= nm \end{aligned}$$

Of course, there are structures for \mathcal{L}_A that have domains other than \mathbb{N} . For instance, we can take \mathfrak{M} with domain $|\mathfrak{M}| = \{a\}^*$ (the finite sequences of the single symbol a , i.e., $\emptyset, a, aa, aaa, \dots$), and interpretations

$$\begin{aligned} o^{\mathfrak{M}} &= \emptyset \\ l^{\mathfrak{M}}(s) &= s \frown a \\ +^{\mathfrak{M}}(n, m) &= a^{n+m} \\ \times^{\mathfrak{M}}(n, m) &= a^{nm} \end{aligned}$$

These two structures are “essentially the same” in the sense that the only difference is the **elements** of the **domains** but not how the **elements** of the **domains** are related among each other by the interpretation functions. We say that the two **structures** are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in \mathfrak{N} also has models that are not isomorphic to \mathfrak{N} . Such structures are called *non-standard*. The interesting thing about them is that while the **elements** of a standard model (i.e., \mathfrak{N} , but also all **structures** isomorphic to it) are exhausted by the values of the standard numerals \bar{n} , i.e.,

$$|\mathfrak{N}| = \{\text{Val}^{\mathfrak{N}}(\bar{n}) : n \in \mathbb{N}\}$$

that isn't the case in non-standard models: if \mathfrak{M} is non-standard, then there is at least one $x \in |\mathfrak{M}|$ such that $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$ for all n .

These non-standard elements are pretty neat: they are “infinite natural numbers.” But their existence also explains, in a sense, the incompleteness phenomena. Consider an example, e.g., the consistency statement for Peano arithmetic, $\text{Con}_{\mathbf{PA}}$, i.e., $\neg \exists x \text{Prf}_{\mathbf{PA}}(x, \ulcorner \perp \urcorner)$. Since \mathbf{PA} neither proves $\text{Con}_{\mathbf{PA}}$ nor $\neg \text{Con}_{\mathbf{PA}}$, either can be consistently added to \mathbf{PA} . Since \mathbf{PA} is consistent, $\mathfrak{N} \models \text{Con}_{\mathbf{PA}}$, and consequently $\mathfrak{N} \not\models \neg \text{Con}_{\mathbf{PA}}$. So \mathfrak{N} is *not* a model of $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$, and all its models must be nonstandard. Models of $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$ must contain some **element** that serves as the witness that makes $\exists x \text{Prf}_{\mathbf{PA}}(\ulcorner \perp \urcorner)$ true, i.e., a Gödel number of a **derivation** of a contradiction from \mathbf{PA} . Such an **element** can't be standard—since $\mathbf{PA} \vdash \neg \text{Prf}_{\mathbf{PA}}(\bar{n}, \ulcorner \perp \urcorner)$ for every n .

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Bibliography