mar.1 Computable Models of Arithmetic

The standard model $\mathfrak{N}$ has two nice features. Its domain is the natural numbers $\mathbb{N}$, i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral $\pi$ actually picks out $n$. The other nice feature is that the interpretations of the non-logical symbols of $\mathcal{L}_A$ are all computable. The successor, addition, and multiplication functions which serve as $\prime^\mathfrak{N}$, $+_\mathfrak{N}$, and $\times^\mathfrak{N}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on $\mathfrak{N}$, i.e., $<^\mathfrak{N}$, is decidable.

Non-standard models of arithmetical theories such as $\mathcal{Q}$ and $\mathcal{PA}$ must contain non-standard elements. Thus their domains typically include elements in addition to $\mathbb{N}$. However, any countable structure can be built on any denumerable set, including $\mathbb{N}$. So there are also non-standard models with domain $\mathbb{N}$. In such models $\mathfrak{M}$, of course, at least some numbers cannot play the roles they usually play, since some $k$ must be different from $\text{Val}^\mathfrak{M}(n)$ for all $n \in \mathbb{N}$.

**Definition mar.1.** A structure $\mathfrak{M}$ for $\mathcal{L}_A$ is computable iff $|\mathfrak{M}| = \mathbb{N}$ and $\prime^\mathfrak{M}$, $+_\mathfrak{M}$, $\times^\mathfrak{M}$ are computable functions and $<^\mathfrak{M}$ is a decidable relation.

**Example mar.2.** Recall the structure $\mathfrak{K}$ from ?? Its domain was $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$\sigma^\mathfrak{K} = 0$

$r^\mathfrak{K}(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases}$

$+_\mathfrak{K}(x, y) = \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$

$\times^\mathfrak{K}(x, y) = \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$

$<^\mathfrak{K} = \{ (x, y) : x, y \in \mathbb{N} \text{ and } x < y \} \cup \{ (x, a) : n \in |\mathfrak{K}| \}$

But $|\mathfrak{K}|$ is denumerable and so is equinumerous with $\mathbb{N}$. For instance, $g: \mathbb{N} \to |\mathfrak{K}|$ with $g(0) = a$ and $g(n) = n + 1$ for $n > 0$ is a bijection. We can turn it into an isomorphism between a new model $\mathcal{R}'$ of $\mathcal{Q}$ and $\mathfrak{K}$. In $\mathcal{R}'$, we have to assign different functions and relations to the symbols of $\mathcal{L}_A$, since different elements of $\mathbb{N}$ play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of $a$, not of the smallest standard number. The smallest standard number is now 1. So we assign $\sigma^{\mathcal{R}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n + 1$. But 0 now plays the role of $a$, which is its own successor. So
\( \rho^{K}(0) = 0. \) For addition and multiplication we likewise have

\[
  +^{K}(x, y) = \begin{cases} 
    x + y & \text{if } x, y > 0 \\
    0 & \text{otherwise}
  \end{cases}
\]

\[
  \times^{K}(x, y) = \begin{cases} 
    xy & \text{if } x, y > 0 \\
    0 & \text{otherwise}
  \end{cases}
\]

And we have \( \langle x, y \rangle \in <^{K} \) iff \( x < y \) and \( x > 0 \) and \( y > 0 \), or if \( y = 0 \).

All of these functions are computable functions of natural numbers and \( <^{K} \) is a decidable relation on \( \mathbb{N} \)—but they are not the same functions as successor, addition, and multiplication on \( \mathbb{N} \), and \( <^{K} \) is not the same relation as \(<\) on \( \mathbb{N} \).

**Problem mar.1.** Give a structure \( L' \) with \( |L'| = \mathbb{N} \) isomorphic to \( L \) of ??.

This example shows that \( \mathbb{Q} \) has computable non-standard models with domain \( \mathbb{N} \). However, the following result shows that this is not true for models of \( \text{PA} \) (and thus also for models of \( \text{TA} \)).

**Theorem mar.3** (Tennenbaum’s Theorem). \( \mathbb{N} \) is the only computable model of \( \text{PA} \).