

mar.1 Computable Models of Arithmetic

The standard model \mathfrak{N} has two nice features. Its domain is the natural numbers \mathbb{N} , i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral \bar{n} actually picks out n . The other nice feature is that the interpretations of the non-logical symbols of \mathcal{L}_A are all *computable*. The successor, addition, and multiplication functions which serve as $r^{\mathfrak{N}}$, $+^{\mathfrak{N}}$, and $\times^{\mathfrak{N}}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on \mathfrak{N} , i.e., $<^{\mathfrak{N}}$, is decidable.

Non-standard models of arithmetical theories such as **Q** and **PA** must contain non-standard elements. Thus their domains typically include *elements* in addition to \mathbb{N} . However, any countable *structure* can be built on any *denumerable* set, including \mathbb{N} . So there are also non-standard models with domain \mathbb{N} . In such models \mathfrak{M} , of course, at least some numbers cannot play the roles they usually play, since some k must be different from $\text{Val}^{\mathfrak{M}}(\bar{n})$ for all $n \in \mathbb{N}$.

Definition mar.1. A structure \mathfrak{M} for \mathcal{L}_A is *computable* iff $|\mathfrak{M}| = \mathbb{N}$ and $r^{\mathfrak{M}}$, $+^{\mathfrak{M}}$, $\times^{\mathfrak{M}}$ are computable functions and $<^{\mathfrak{M}}$ is a decidable relation.

Example mar.2. Recall the structure \mathfrak{K} from ?? Its domain was $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$$\begin{aligned} o^{\mathfrak{K}} &= 0 \\ r^{\mathfrak{K}}(x) &= \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\} \end{aligned}$$

But $|\mathfrak{K}|$ is *denumerable* and so is equinumerous with \mathbb{N} . For instance, $g: \mathbb{N} \rightarrow |\mathfrak{K}|$ with $g(0) = a$ and $g(n) = n + 1$ for $n > 0$ is a *bijection*. We can turn it into an isomorphism between a new model \mathfrak{K}' of **Q** and \mathfrak{K} . In \mathfrak{K}' , we have to assign different functions and relations to the symbols of \mathcal{L}_A , since different *elements* of \mathbb{N} play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of a , not of the smallest standard number. The smallest standard number is now 1. So we assign $o^{\mathfrak{K}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n + 1$. But 0 now plays the role of a , which is its own successor. So

$\neq^{\mathfrak{K}'}(0) = 0$. For addition and multiplication we likewise have

$$+^{\mathfrak{K}'}(x, y) = \begin{cases} x + y & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\times^{\mathfrak{K}'}(x, y) = \begin{cases} xy & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

And we have $\langle x, y \rangle \in <^{\mathfrak{K}'}$ iff $x < y$ and $x > 0$ and $y > 0$, or if $y = 0$.

All of these functions are computable functions of natural numbers and $<^{\mathfrak{K}'}$ is a decidable relation on \mathbb{N} —but they are not the same functions as successor, addition, and multiplication on \mathbb{N} , and $<^{\mathfrak{K}'}$ is not the same relation as $<$ on \mathbb{N} .

Problem mar.1. Give a structure \mathfrak{L}' with $|\mathfrak{L}'| = \mathbb{N}$ isomorphic to \mathfrak{L} of ??.

explanation

This example shows that \mathbf{Q} has computable non-standard models with domain \mathbb{N} . However, the following result shows that this is not true for models of \mathbf{PA} (and thus also for models of \mathbf{TA}).

Theorem mar.3 (Tennenbaum's Theorem). *\mathfrak{N} is the only computable model of \mathbf{PA} .*

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Bibliography