mar.1  Computable Models of Arithmetic

The standard model $\mathcal{N}$ has two nice features. Its domain is the natural numbers $\mathbb{N}$, i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral $\pi$ actually picks out $n$. The other nice feature is that the interpretations of the non-logical symbols of $\mathcal{L}_A$ are all computable. The successor, addition, and multiplication functions which serve as $\prime^{\mathcal{N}}$, $+^{\mathcal{N}}$, and $\times^{\mathcal{N}}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on $\mathcal{N}$, i.e., $<^{\mathcal{N}}$, is decidable.

Non-standard models of arithmetical theories such as $\mathbb{Q}$ and $\mathbb{PA}$ must contain non-standard elements. Thus their domains typically include elements in addition to $\mathbb{N}$. However, any countable structure can be built on any denumerable set, including $\mathbb{N}$. So there are also non-standard models with domain $\mathbb{N}$. In such models $\mathcal{M}$, of course, at least some numbers cannot play the roles they usually play, since some $k$ must be different from $\text{Val}(\mathcal{M})(n)$ for all $n \in \mathbb{N}$.

**Definition mar.1.** A structure $\mathcal{M}$ for $\mathcal{L}_A$ is computable iff $|\mathcal{M}| = \mathbb{N}$ and $\prime^{\mathcal{M}}$, $+^{\mathcal{M}}$, $\times^{\mathcal{M}}$ are computable functions and $<^{\mathcal{M}}$ is a decidable relation.

**Example mar.2.** Recall the structure $\mathfrak{R}$ from ?? Its domain was $|\mathfrak{R}| = \mathbb{N} \cup \{a\}$ and interpretations

\[
\begin{align*}
\sigma^{\mathfrak{R}} &= 0 \\
\prime^{\mathfrak{R}}(x) &= \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\
+^{\mathfrak{R}}(x, y) &= \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
\times^{\mathfrak{R}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
<^{\mathfrak{R}} &= \{(x, y) : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{(x, a) : n \in |\mathfrak{R}|\}
\end{align*}
\]

But $|\mathfrak{R}|$ is denumerable and so is equinumerous with $\mathbb{N}$. For instance, $g: \mathbb{N} \to |\mathfrak{R}|$ with $g(0) = a$ and $g(n) = n + 1$ for $n > 0$ is a bijection. We can turn it into an isomorphism between a new model $\mathcal{R}'$ of $\mathbb{Q}$ and $\mathfrak{R}$. In $\mathcal{R}'$, we have to assign different functions and relations to the symbols of $\mathcal{L}_A$, since different elements of $\mathbb{N}$ play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of $a$, not of the smallest standard number. The smallest standard number is now 1. So we assign $\sigma^{\mathcal{R}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n + 1$. But 0 now plays the role of $a$, which is its own successor. So
$\rho^R(0) = 0$. For addition and multiplication we likewise have

$$
+^R(x, y) = \begin{cases} 
  x + y & \text{if } x, y > 0 \\
  0 & \text{otherwise}
\end{cases}
$$

$$
\times^R(x, y) = \begin{cases} 
  xy & \text{if } x, y > 0 \\
  0 & \text{otherwise}
\end{cases}
$$

And we have $\langle x, y \rangle \in <^R$ iff $x < y$ and $x > 0$ and $y > 0$, or if $y = 0$.

All of these functions are computable functions of natural numbers and $<^R$ is a decidable relation on $\mathbb{N}$—but they are not the same functions as successor, addition, and multiplication on $\mathbb{N}$, and $<^R$ is not the same relation as $<$ on $\mathbb{N}$.

**Problem mar.1.** Give a structure $\mathcal{L}'$ with $|\mathcal{L}'| = \mathbb{N}$ isomorphic to $\mathcal{L}$ of ??.

**Explanation**
This example shows that $\mathbb{Q}$ has computable non-standard models with domain $\mathbb{N}$. However, the following result shows that this is not true for models of PA (and thus also for models of TA).

**Theorem mar.3** (Tennenbaum’s Theorem). $\mathbb{R}$ is the only computable model of PA.

**Photo Credits**

**Bibliography**