mar.1  Computable Models of Arithmetic

The standard model $\mathfrak{N}$ has two nice features. Its domain is the natural numbers $\mathbb{N}$, i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral $n$ actually picks out $n$. The other nice feature is that the interpretations of the non-logical symbols of $L_A$ are all *computable*. The successor, addition, and multiplication functions which serve as $\cdot^{\mathfrak{N}}$, $+^{\mathfrak{N}}$, and $\times^{\mathfrak{N}}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on $\mathfrak{N}$, i.e., $<^{\mathfrak{N}}$, is decidable.

Non-standard models of arithmetical theories such as $\mathbb{Q}$ and $\mathbf{PA}$ must contain non-standard elements. Thus their domains typically include elements in addition to $\mathbb{N}$. However, any countable structure can be built on any denumerable set, including $\mathbb{N}$. So there are also non-standard models with domain $\mathbb{N}$. In such models $\mathfrak{M}$, of course, at least some numbers cannot play the roles they usually play, since some $k$ must be different from $\text{Val}^{\mathfrak{M}}(n)$ for all $n \in \mathbb{N}$.

**Definition mar.1.** A structure $\mathfrak{M}$ for $L_A$ is *computable* iff $|\mathfrak{M}| = \mathbb{N}$ and $\cdot^{\mathfrak{M}}$, $+^{\mathfrak{M}}$, $\times^{\mathfrak{M}}$ are computable functions and $<^{\mathfrak{M}}$ is a decidable relation.

**Example mar.2.** Recall the structure $\mathfrak{K}$ from ???. Its domain was $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$$
\begin{align*}
\sigma^{\mathfrak{K}} &= 0 \\
\cdot^{\mathfrak{K}}(x) &= \begin{cases} 
  x + 1 & \text{if } x \in \mathbb{N} \\
  a & \text{if } x = a 
\end{cases} \\
+^{\mathfrak{K}}(x, y) &= \begin{cases} 
  x + y & \text{if } x, y \in \mathbb{N} \\
  a & \text{otherwise} 
\end{cases} \\
\times^{\mathfrak{K}}(x, y) &= \begin{cases} 
  xy & \text{if } x, y \in \mathbb{N} \\
  0 & \text{if } x = 0 \text{ or } y = 0 \\
  a & \text{otherwise} 
\end{cases} \\
<^{\mathfrak{K}} &= \{ (x, y) : x, y \in \mathbb{N} \text{ and } x < y \} \cup \{ (x, a) : n \in |\mathfrak{K}| \}
\end{align*}
$$

But $|\mathfrak{K}|$ is *denumerable* and so is equinumerous with $\mathbb{N}$. For instance, $g : \mathbb{N} \to |\mathfrak{K}|$ with $g(0) = a$ and $g(n) = n + 1$ for $n > 0$ is a bijection. We can turn it into an isomorphism between a new model $\mathfrak{R}'$ of $\mathbb{Q}$ and $\mathfrak{K}$. In $\mathfrak{R}'$, we have to assign different functions and relations to the symbols of $L_A$, since different elements of $\mathbb{N}$ play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of $a$, not of the smallest standard number. The smallest standard number is now 1. So we assign $\sigma^{\mathfrak{R}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n + 1$. But 0 now plays the role of $a$, which is its own successor. So
\( \sigma'(0) = 0 \). For addition and multiplication we likewise have

\[
\begin{align*}
+_{\mathcal{K}'}(x, y) &= \begin{cases} 
  x + y - 1 & \text{if } x, y > 0 \\
  0 & \text{otherwise}
\end{cases} \\
\times_{\mathcal{K}'}(x, y) &= \begin{cases} 
  1 & \text{if } x = 1 \text{ or } y = 1 \\
  xy - x - y + 2 & \text{if } x, y > 1 \\
  0 & \text{otherwise}
\end{cases}
\end{align*}
\]

And we have \( \langle x, y \rangle \in <_{\mathcal{K}'} \) iff \( x < y \) and \( x > 0 \) and \( y > 0 \), or if \( y = 0 \).

All of these functions are computable functions of natural numbers and \( <_{\mathcal{K}'} \) is a decidable relation on \( \mathbb{N} \) — but they are not the same functions as successor, addition, and multiplication on \( \mathbb{N} \), and \( <_{\mathcal{K}'} \) is not the same relation as \( < \) on \( \mathbb{N} \).

**Problem mar.1.** Give a structure \( \mathcal{L}' \) with \( |\mathcal{L}'| = \mathbb{N} \) isomorphic to \( \mathcal{L} \) of ??.

**Example mar.2** shows that \( \mathbb{Q} \) has computable non-standard models with domain \( \mathbb{N} \). However, the following result shows that this is not true for models of \( \text{PA} \) (and thus also for models of \( \text{TA} \)).

**Theorem mar.3 (Tennenbaum’s Theorem).** \( \mathbb{N} \) is the only computable model of \( \text{PA} \).

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**Bibliography**