mar.1  Computable Models of Arithmetic

The standard model $\mathcal{M}$ has two nice features. Its domain is the natural numbers $\mathbb{N}$, i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral $n$ actually picks out $n$. The other nice feature is that the interpretations of the non-logical symbols of $\mathcal{L}_A$ are all computable. The successor, addition, and multiplication functions which serve as $\nu^\mathcal{M}$, $+^\mathcal{M}$, and $\times^\mathcal{M}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on $\mathcal{M}$, i.e., $<^\mathcal{M}$, is decidable.

Non-standard models of arithmetical theories such as $Q$ and $PA$ must contain non-standard elements. Thus their domains typically include elements in addition to $\mathbb{N}$. However, any countable structure can be built on any denumerable set, including $\mathbb{N}$. So there are also non-standard models with domain $\mathbb{N}$. In such models $\mathcal{M}$, of course, at least some numbers cannot play the roles they usually play, since some $k$ must be different from $Val^\mathcal{M}(n)$ for all $n \in \mathbb{N}$.

Definition mar.1. A structure $\mathcal{M}$ for $\mathcal{L}_A$ is computable iff $|\mathcal{M}| = \mathbb{N}$ and $\nu^\mathcal{M}$, $+^\mathcal{M}$, $\times^\mathcal{M}$ are computable functions and $<^\mathcal{M}$ is a decidable relation.

Example mar.2. Recall the structure $\mathcal{R}$ from ?? Its domain was $|\mathcal{R}| = \mathbb{N} \cup \{a\}$ and interpretations

$$o^\mathcal{R} = 0$$
$$\nu^\mathcal{R}(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases}$$
$$+^\mathcal{R}(x, y) = \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$$
$$\times^\mathcal{R}(x, y) = \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$$
$$<^\mathcal{R} = \{(x, y) : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{(x, a) : n \in |\mathcal{R}|\}$$

But $|\mathcal{R}|$ is denumerable and so is equinumerous with $\mathbb{N}$. For instance, $g: \mathbb{N} \to |\mathcal{R}|$ with $g(0) = a$ and $g(n) = n + 1$ for $n > 0$ is a bijection. We can turn it into an isomorphism between a new model $\mathcal{R}'$ of $Q$ and $\mathcal{R}$. In $\mathcal{R}'$, we have to assign different functions and relations to the symbols of $\mathcal{L}_A$, since different elements of $\mathbb{N}$ play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of $a$, not of the smallest standard number. The smallest standard number is now 1. So we assign $o^{\mathcal{R}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n + 1$. But 0 now plays the role of $a$, which is its own successor. So
\( \rho'(0) = 0 \). For addition and multiplication we likewise have

\[
+^{\mathcal{K}'}(x, y) = \begin{cases} 
  x + y & \text{if } x, y > 0 \\
  0 & \text{otherwise}
\end{cases}
\]

\[
\times^{\mathcal{K}'}(x, y) = \begin{cases} 
  xy & \text{if } x, y > 0 \\
  0 & \text{otherwise}
\end{cases}
\]

And we have \( \langle x, y \rangle \in <^{\mathcal{K}'} \) iff \( x < y \) and \( x > 0 \) and \( y > 0 \), or if \( y = 0 \).

All of these functions are computable functions of natural numbers and \( <^{\mathcal{K}'} \) is a decidable relation on \( \mathbb{N} \)—but they are not the same functions as successor, addition, and multiplication on \( \mathbb{N} \), and \( <^{\mathcal{K}'} \) is not the same relation as \( < \) on \( \mathbb{N} \).

**Problem mar.1.** Give a structure \( \mathcal{L}' \) with \( |\mathcal{L}'| = \mathbb{N} \) isomorphic to \( \mathcal{L} \) of ??.

This example shows that \( \mathbb{Q} \) has computable non-standard models with domain \( \mathbb{N} \). However, the following result shows that this is not true for models of \( \text{PA} \) (and thus also for models of \( \text{TA} \)).

**Theorem mar.3 (Tennenbaum’s Theorem).** \( \mathcal{N} \) is the only computable model of \( \text{PA} \).

**Photo Credits**

**Bibliography**