Part I

Model Theory

Material on model theory is incomplete and experimental. It is currently simply an adaptation of Aldo Antonelli's notes on model theory, less those topics covered in the part on first-order logic (theories, completeness, compactness). It requires much more introduction, motivation, and explanation, as well as exercises, to be useful for a textbook. Andy Arana is at planning to work on this part specifically (issue #65).

Chapter 1

Basics of Model Theory

1.1 Reducts and Expansions

Often it is useful or necessary to compare languages which have symbols in common, as well as structures for these languages. The most common case is when all the symbols in a language $\mathcal{L}$ are also part of a language $\mathcal{L}'$, i.e., $\mathcal{L} \subseteq \mathcal{L}'$. An $\mathcal{L}$-structure $\mathfrak{M}$ can then always be expanded to an $\mathcal{L}'$-structure by adding interpretations of the additional symbols while leaving the interpretations of the common symbols the same. On the other hand, from an $\mathcal{L}'$-structure $\mathfrak{M}'$ we can obtain an $\mathcal{L}$-structure simply by "forgetting" the interpretations of the symbols that do not occur in $\mathcal{L}$.

mod:bas:red: defn:reduct

Definition 1.1. Suppose $\mathcal{L} \subseteq \mathcal{L}'$, $\mathfrak{M}$ is an $\mathcal{L}$-structure and $\mathfrak{M}'$ is an $\mathcal{L}'$-structure. $\mathfrak{M}$ is the *reduct* of $\mathfrak{M}'$ to $\mathcal{L}$, and $\mathfrak{M}'$ is an *expansion* of $\mathfrak{M}$ to $\mathcal{L}'$ iff

1. $|\mathfrak{M}| = |\mathfrak{M}'|$
2. For every constant symbol $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$.
3. For every function symbol $f \in \mathcal{L}$, $f^{\mathfrak{M}} = f^{\mathfrak{M}'}$.
4. For every predicate symbol $P \in \mathcal{L}$, $P^{\mathfrak{M}} = P^{\mathfrak{M}'}$.

mod:bas:red: prop:reduct

Proposition 1.2. *If an $\mathcal{L}$-structure $\mathfrak{M}$ is a reduct of an $\mathcal{L}'$-structure $\mathfrak{M}'$, then for all $\mathcal{L}$-sentences φ,*

$$\mathfrak{M} \vDash \varphi \text{ iff } \mathfrak{M}' \vDash \varphi.$$

Proof. Exercise. □

Problem 1.1. Prove Proposition 1.2.

Definition 1.3. When we have an $\mathcal{L}$-structure $\mathfrak{M}$, and $\mathcal{L}' = \mathcal{L} \cup \{P\}$ is the expansion of $\mathcal{L}$ obtained by adding a single n-place predicate symbol P, and $R \subseteq |\mathfrak{M}|^n$ is an n-place relation, then we write $(\mathfrak{M}, R)$ for the expansion $\mathfrak{M}'$ of $\mathfrak{M}$ with $P^{\mathfrak{M}'} = R$.

1.2 Substructures

The domain of a structure $\mathfrak{M}$ may be a subset of another $\mathfrak{M}'$. But we should obviously only consider $\mathfrak{M}$ a "part" of $\mathfrak{M}'$ if not only $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$, but $\mathfrak{M}$ and $\mathfrak{M}'$ "agree" in how they interpret the symbols of the language at least on the shared part $|\mathfrak{M}|$. mod:bas:sub:sec

Definition 1.4. Given structures $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is a *substructure* of $\mathfrak{M}'$, and $\mathfrak{M}'$ an *extension* of $\mathfrak{M}$, written $\mathfrak{M} \subseteq \mathfrak{M}'$, iff mod:bas:sub:defn:substructure

1. $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$,
2. For each constant $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$;
3. For each n-place function symbol $f \in \mathcal{L}$ $f^{\mathfrak{M}}(a_1, \dots, a_n) = f^{\mathfrak{M}'}(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in |\mathfrak{M}|$.
4. For each n-place predicate symbol $R \in \mathcal{L}$, $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{M}}$ iff $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{M}'}$ for all $a_1, \dots, a_n \in |\mathfrak{M}|$.

Remark 1. If the language contains no constant or function symbols, then any $N \subseteq |\mathfrak{M}|$ determines a substructure $\mathfrak{N}$ of $\mathfrak{M}$ with domain $|\mathfrak{N}| = N$ by putting $R^{\mathfrak{N}} = R^{\mathfrak{M}} \cap N^n$. mod:bas:sub:rem:substructure

1.3 Overspill

mod:bas:ove:sec

Theorem 1.5. *If a set Γ of sentences has arbitrarily large finite models, then it has an infinite model.* *mod:bas:ove:overspill*

Proof. Expand the language of Γ by adding countably many new constants c_0, c_1, …and consider the set $\Gamma \cup \{c_i \neq c_j : i \neq j\}$. To say that Γ has arbitrarily large finite models means that for every $m > 0$ there is $n \geq m$ such that Γ has a model of cardinality n. This implies that $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ is finitely satisfiable. By compactness, $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ has a model $\mathfrak{M}$ whose domain must be infinite, since it satisfies all inequalities $c_i \neq c_j$. □

Proposition 1.6. *There is no sentence φ of any first-order language that is true in a structure $\mathfrak{M}$ if and only if the domain $|\mathfrak{M}|$ of the structure is infinite.* *mod:bas:ove:inf-not-fo*

Proof. If there were such a φ, its negation $\neg\varphi$ would be true in all and only the finite structures, and it would therefore have arbitrarily large finite models but it would lack an infinite model, contradicting Theorem 1.5. □

1.4 Isomorphic Structures

mod:bas:iso:sec

First-order structures can be alike in one of two ways. One way in which the can be alike is that they make the same sentences true. We call such structures *elementarily equivalent*. But structures can be very different and still make the same sentences true—for instance, one can be enumerable and the other not. This is because there are lots of features of a structure that cannot be expressed in first-order languages, either because the language is not rich enough, or because of fundamental limitations of first-order logic such as the Löwenheim–Skolem theorem. So another, stricter, aspect in which structures can be alike is if they are fundamentally the same, in the sense that they only differ in the objects that make them up, but not in their structural features. A way of making this precise is by the notion of an *isomorphism*.

mod:bas:iso:defn:elem-equiv

Definition 1.7. Given two structures $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is *elementarily equivalent to* $\mathfrak{M}'$, written $\mathfrak{M} \equiv \mathfrak{M}'$, if and only if for every sentence φ of $\mathcal{L}$, $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$.

mod:bas:iso:defn:isomorphism

Definition 1.8. Given two structures $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is *isomorphic to* $\mathfrak{M}'$, written $\mathfrak{M} \simeq \mathfrak{M}'$, if and only if there is a function $h\colon |\mathfrak{M}| \to |\mathfrak{M}'|$ such that:

1. h is injective: if $h(x) = h(y)$ then $x = y$;
2. h is surjective: for every $y \in |\mathfrak{M}'|$ there is $x \in |\mathfrak{M}|$ such that $h(x) = y$;
3. mod:bas:iso:defn:iso-const for every constant symbol c: $h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$;
4. mod:bas:iso:defn:iso-pred for every n-place predicate symbol P:
$$\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}} \quad \text{iff} \quad \langle h(a_1), \dots, h(a_n) \rangle \in P^{\mathfrak{M}'};$$
5. mod:bas:iso:defn:iso-func for every n-place function symbol f:
$$h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{M}'}(h(a_1), \dots, h(a_n)).$$

mod:bas:iso:thm:isom

Theorem 1.9. *If $\mathfrak{M} \simeq \mathfrak{M}'$ then $\mathfrak{M} \equiv \mathfrak{M}'$.*

Proof. Let h be an isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$. For any assignment s, $h \circ s$ is the composition of h and s, i.e., the assignment in $\mathfrak{M}'$ such that $(h \circ s)(x) = h(s(x))$. By induction on t and φ one can prove the stronger claims:

a. $h(\mathrm{Val}^{\mathfrak{M}}_s(t)) = \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t)$.

b. $\mathfrak{M}, s \vDash \varphi$ iff $\mathfrak{M}', h \circ s \vDash \varphi$.

The first is proved by induction on the complexity of t.

1. If $t \equiv c$, then $\mathrm{Val}^{\mathfrak{M}}_s(c) = c^{\mathfrak{M}}$ and $\mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(c) = c^{\mathfrak{M}'}$. Thus, $h(\mathrm{Val}^{\mathfrak{M}}_s(t)) = h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ (by (3) of Definition 1.8) $= \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t)$.

2. If $t \equiv x$, then $\mathrm{Val}_s^{\mathfrak{M}}(x) = s(x)$ and $\mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(x) = h(s(x))$. Thus, $h(\mathrm{Val}_s^{\mathfrak{M}}(x)) = h(s(x)) = \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(x)$.

3. If $t \equiv f(t_1, \dots, t_n)$, then

$$\begin{aligned}\mathrm{Val}_s^{\mathfrak{M}}(t) &= f^{\mathfrak{M}}(\mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)) \quad \text{and} \\ \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t) &= f^{\mathfrak{M}}(\mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t_1), \dots, \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t_n)).\end{aligned}$$

The induction hypothesis is that for each i, $h(\mathrm{Val}_s^{\mathfrak{M}}(t_i)) = \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t_i)$. So,

$$\begin{aligned}h(\mathrm{Val}_s^{\mathfrak{M}}(t)) &= h(f^{\mathfrak{M}}(\mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)) \\ &= h(f^{\mathfrak{M}}(\mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t_1), \dots, \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t_n)) \qquad (1.1) \\ &= f^{\mathfrak{M}'}(\mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t_1), \dots, \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t_n)) \qquad (1.2) \\ &= \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t)\end{aligned}$$

mod:bas:iso:iso-1
mod:bas:iso:iso-2

Here, eq. (1.1) follows by induction hypothesis and eq. (1.2) by (5) of Definition 1.8.

Part (b) is left as an exercise.

If φ is a sentence, the assignments s and $h \circ s$ are irrelevant, and we have $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$. □

Problem 1.2. Carry out the proof of (b) of Theorem 1.9 in detail. Make sure to note where each of the five properties characterizing isomorphisms of Definition 1.8 is used.

Definition 1.10. An *automorphism* of a structure $\mathfrak{M}$ is an isomorphism of $\mathfrak{M}$ onto itself.

Problem 1.3. Show that for any structure $\mathfrak{M}$, if X is a definable subset of $\mathfrak{M}$, and h is an automorphism of $\mathfrak{M}$, then $X = \{h(x) : x \in X\}$ (i.e., X is fixed under h).

1.5 The Theory of a Structure

Every structure $\mathfrak{M}$ makes some sentences true, and some false. The set of all the sentences it makes true is called its *theory*. That set is in fact a theory, since anything it entails must be true in all its models, including $\mathfrak{M}$.

Definition 1.11. Given a structure $\mathfrak{M}$, the *theory* of $\mathfrak{M}$ is the set $\mathrm{Th}(\mathfrak{M})$ of sentences that are true in $\mathfrak{M}$, i.e., $\mathrm{Th}(\mathfrak{M}) = \{\varphi : \mathfrak{M} \vDash \varphi\}$.

We also use the term "theory" informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

Proposition 1.12. *For any $\mathfrak{M}$, $\mathrm{Th}(\mathfrak{M})$ is complete.*

Proof. For any sentence φ either $\mathfrak{M} \models \varphi$ or $\mathfrak{M} \models \neg\varphi$, so either $\varphi \in \mathrm{Th}(\mathfrak{M})$ or $\neg\varphi \in \mathrm{Th}(\mathfrak{M})$. □

mod:bas:thm:prop:equiv

Proposition 1.13. *If $\mathfrak{N} \models \varphi$ for every $\varphi \in \mathrm{Th}(\mathfrak{M})$, then $\mathfrak{M} \equiv \mathfrak{N}$.*

Proof. Since $\mathfrak{N} \models \varphi$ for all $\varphi \in \mathrm{Th}(\mathfrak{M})$, $\mathrm{Th}(\mathfrak{M}) \subseteq \mathrm{Th}(\mathfrak{N})$. If $\mathfrak{N} \models \varphi$, then $\mathfrak{N} \not\models \neg\varphi$, so $\neg\varphi \notin \mathrm{Th}(\mathfrak{M})$. Since $\mathrm{Th}(\mathfrak{M})$ is complete, $\varphi \in \mathrm{Th}(\mathfrak{M})$. So, $\mathrm{Th}(\mathfrak{N}) \subseteq \mathrm{Th}(\mathfrak{M})$, and we have $\mathfrak{M} \equiv \mathfrak{N}$. □

mod:bas:thm:remark:R

Remark 2. Consider $\mathfrak{R} = \langle \mathbb{R}, < \rangle$, the structure whose domain is the set $\mathbb{R}$ of the real numbers, in the language comprising only a 2-place predicate symbol interpreted as the $<$ relation over the reals. Clearly $\mathfrak{R}$ is non-enumerable; however, since $\mathrm{Th}(\mathfrak{R})$ is obviously consistent, by the Löwenheim–Skolem theorem it has an enumerable model, say $\mathfrak{S}$, and by Proposition 1.13, $\mathfrak{R} \equiv \mathfrak{S}$. Moreover, since $\mathfrak{R}$ and $\mathfrak{S}$ are not isomorphic, this shows that the converse of Theorem 1.9 fails in general.

1.6 Partial Isomorphisms

Definition 1.14. Given two structures $\mathfrak{M}$ and $\mathfrak{N}$, a *partial isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ is a finite partial function p taking arguments in $|\mathfrak{M}|$ and returning values in $|\mathfrak{N}|$, which satisfies the isomorphism conditions from Definition 1.8 on its domain:

1. p is injective;
2. for every constant symbol c: if $p(c^{\mathfrak{M}})$ is defined, then $p(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$;
3. for every n-place predicate symbol P: if $a_1, \dots, a_n$ are in the domain of p, then $\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}}$ if and only if $\langle p(a_1), \dots, p(a_n) \rangle \in P^{\mathfrak{N}}$;
4. for every n-place function symbol f: if $a_1, \dots, a_n$ are in the domain of p, then $p(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{N}}(p(a_1), \ dots, p(a_n))$.

That p is finite means that $\mathrm{dom}(p)$ is finite.

Notice that the empty function $\emptyset$ is always a partial isomorphism between any two structures.

mod:bas:pis:defn:partialisom

Definition 1.15. Two structures $\mathfrak{M}$ and $\mathfrak{N}$, are *partially isomorphic*, written $\mathfrak{M} \simeq_p \mathfrak{N}$, if and only if there is a non-empty set I of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ satisfying the *back-and-forth* property:

1. (*Forth*) For every $p \in I$ and $a \in |\mathfrak{M}|$ there is $q \in I$ such that $p \subseteq q$ and a is in the domain of q;
2. (*Back*) For every $p \in I$ and $b \in |\mathfrak{N}|$ there is $q \in I$ such that $p \subseteq q$ and b is in the range of q.

Theorem 1.16. *If $\mathfrak{M} \simeq_p \mathfrak{N}$ and $\mathfrak{M}$ and $\mathfrak{N}$ are enumerable, then $\mathfrak{M} \simeq \mathfrak{N}$.* mod:bas:pis: thm:p-isom1

Proof. Since $\mathfrak{M}$ and $\mathfrak{N}$ are enumerable, let $|\mathfrak{M}| = \{a_0, a_1, \ldots\}$ and $|\mathfrak{N}| = \{b_0, b_1, \ldots\}$. Starting with an arbitrary $p_0 \in I$, we define an increasing sequence of partial isomorphisms $p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$ as follows:

1. if $n+1$ is odd, say $n = 2r$, then using the Forth property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and a_r is in the domain of p_{n+1};
2. if $n+1$ is even, say $n+1 = 2r$, then using the Back property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and b_r is in the range of p_{n+1}.

If we now put:

$$p = \bigcup_{n \geq 0} p_n,$$

we have that p is a an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. □

Problem 1.4. Show in detail that p as defined in Theorem 1.16 is in fact an isomorphism.

Theorem 1.17. *Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are structures for a purely relational language (a language containing only predicate symbols, and no function symbols or constants). Then if $\mathfrak{M} \simeq_p \mathfrak{N}$, also $\mathfrak{M} \equiv \mathfrak{N}$.* mod:bas:pis: thm:p-isom2

Proof. By induction on formulas, one shows that if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are such that there is a partial isomorphism p mapping each a_i to b_i and $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ (for $i = 1, \ldots, n$), then $\mathfrak{M}, s_1 \vDash \varphi$ if and only if $\mathfrak{N}, s_2 \vDash \varphi$. The case for $n = 0$ gives $\mathfrak{M} \equiv \mathfrak{N}$. □

Remark 3. If function symbols are present, the previous result is still true, but one needs to consider the isomorphism induced by p between the substructure of $\mathfrak{M}$ generated by $a_1, \ldots, a_n$ and the substructure of $\mathfrak{N}$ generated by $b_1, \ldots, b_n$.

The previous result can be "broken down" into stages by establishing a connection between the number of nested quantifiers in a formula and how many times the relevant partial isomorphisms can be extended.

Definition 1.18. For any formula φ, the *quantifier rank* of φ, denoted by $\mathrm{qr}(\varphi) \in \mathbb{N}$, is recursively defined as the highest number of nested quantifiers in φ. Two structures $\mathfrak{M}$ and $\mathfrak{N}$ are *n-equivalent*, written $\mathfrak{M} \equiv_n \mathfrak{N}$, if they agree on all sentences of quantifier rank less than or equal to n.

Proposition 1.19. *Let $\mathcal{L}$ be a finite purely relational language, i.e., a language containing finitely many predicate symbols and constant symbols, and no function symbols. Then for each $n \in \mathbb{N}$ there are only finitely many first-order sentences in the language $\mathcal{L}$ that have quantifier rank no greater than n, up to logical equivalence.* mod:bas:pis: prop:qr-finite

Proof. By induction on n. □

Definition 1.20. Given a structure $\mathfrak{M}$, let $|\mathfrak{M}|^{<\omega}$ be the set of all finite sequences over $|\mathfrak{M}|$. We use $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ to range over finite sequences of elements. If $\mathbf{a} \in |\mathfrak{M}|^{<\omega}$ and $a \in |\mathfrak{M}|$, then $\mathbf{a}a$ represents the *concatenation* of $\mathbf{a}$ with a.

Definition 1.21. Given structures $\mathfrak{M}$ and $\mathfrak{N}$, we define relations $I_n \subseteq |\mathfrak{M}|^{<\omega} \times |\mathfrak{N}|^{<\omega}$ between sequences of equal length, by recursion on n as follows:

1. $I_0(\mathbf{a}, \mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ satisfy the same atomic formulas in $\mathfrak{M}$ and $\mathfrak{N}$; i.e., if $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ and φ is atomic with all variables among $x_1, \dots, x_n$, then $\mathfrak{M}, s_1 \vDash \varphi$ if and only if $\mathfrak{N}, s_2 \vDash \varphi$.

2. $I_{n+1}(\mathbf{a}, \mathbf{b})$ if and only if for every $a \in A$ there is a $b \in B$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and vice-versa.

Definition 1.22. Write $\mathfrak{M} \approx_n \mathfrak{N}$ if $I_n(\Lambda, \Lambda)$ holds of $\mathfrak{M}$ and $\mathfrak{N}$ (where Λ is the empty sequence).

mod:bas:pis:thm:b-n-f **Theorem 1.23.** *Let $\mathcal{L}$ be a purely relational language. Then $I_n(\mathbf{a}, \mathbf{b})$ implies that for every φ such that $\operatorname{qr}(\varphi) \leq n$, we have $\mathfrak{M}, \mathbf{a} \vDash \varphi$ if and only if $\mathfrak{N}, \mathbf{b} \vDash \varphi$ (where again $\mathbf{a}$ satisfies φ if any s such that $s(x_i) = a_i$ satisfies φ). Moreover, if $\mathcal{L}$ is finite, the converse also holds.*

Proof. The proof that $I_n(\mathbf{a}, \mathbf{b})$ implies that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of quantifier rank no greater than n is by an easy induction on φ. For the converse we proceed by induction on n, using Proposition 1.19, which ensures that for each n there are at most finitely many non-equivalent formulas of that quantifier rank.

For $n = 0$ the hypothesis that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same quantifier-free formulas gives that they satisfy the same atomic ones, so that $I_0(\mathbf{a}, \mathbf{b})$.

For the $n + 1$ case, suppose that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of quantifier rank no greater than $n + 1$; in order to show that $I_{n+1}(\mathbf{a}, \mathbf{b})$ suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and by the inductive hypothesis again suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n.

Given $a \in |\mathfrak{M}|$, let τ_n^a be set of formulas $\psi(x, \mathbf{y})$ of rank no greater than n satisfied by $\mathbf{a}a$ in $\mathfrak{M}$; τ_n^a is finite, so we can assume it is a single first-order formula. It follows that $\mathbf{a}$ satisfies $\exists x\, \tau_n^a(x, \mathbf{y})$, which has quantifier rank no greater than $n + 1$. By hypothesis $\mathbf{b}$ satisfies the same formula in $\mathfrak{N}$, so that there is a $b \in |\mathfrak{N}|$ such that $\mathbf{b}b$ satisfies τ_n^a; in particular, $\mathbf{b}b$ satisfies the same formulas of quantifier rank no greater than n as $\mathbf{a}a$. Similarly one shows that for every $b \in |\mathfrak{N}|$ there is $a \in |\mathfrak{M}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n, which completes the proof. □

mod:bas:pis:cor:b-n-f **Corollary 1.24.** *If $\mathfrak{M}$ and $\mathfrak{N}$ are purely relational structures in a finite language, then $\mathfrak{M} \approx_n \mathfrak{N}$ if and only if $\mathfrak{M} \equiv_n \mathfrak{N}$. In particular $\mathfrak{M} \equiv \mathfrak{N}$ if and only if for each n, $\mathfrak{M} \approx_n \mathfrak{N}$.*

1.7 Dense Linear Orders

Definition 1.25. A *dense linear ordering without endpoints* is a structure $\mathfrak{M}$ for the language containing a single 2-place predicate symbol $<$ satisfying the following sentences:

1. $\forall x\, \neg x < x$;
2. $\forall x\, \forall y\, \forall z\, (x < y \rightarrow (y < z \rightarrow x < z))$;
3. $\forall x\, \forall y\, (x < y \lor x = y \lor y < x)$;
4. $\forall x\, \exists y\, x < y$;
5. $\forall x\, \exists y\, y < x$;
6. $\forall x\, \forall y\, (x < y \rightarrow \exists z\, (x < z \land z < y))$.

Theorem 1.26. *Any two enumerable dense linear orderings without endpoints are isomorphic.* mod:bas:dlo:thm:cantorQ

Proof. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be enumerable dense linear orderings without endpoints, with $<_1 = <^{\mathfrak{M}_1}$ and $<_2 = <^{\mathfrak{M}_2}$, and let $\mathcal{I}$ be the set of all partial isomorphisms between them. $\mathcal{I}$ is not empty since at least $\emptyset \in \mathcal{I}$. We show that $\mathcal{I}$ satisfies the Back-and-Forth property. Then $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$, and the theorem follows by Theorem 1.16.

To show $\mathcal{I}$ satisfies the Forth property, let $p \in \mathcal{I}$ and let $p(a_i) = b_i$ for $i = 1$, …, n, and without loss of generality suppose $a_1 <_1 a_2 <_1 \cdots <_1 a_n$. Given $a \in |\mathfrak{M}_1|$, find $b \in |\mathfrak{M}_2|$ as follows:

1. if $a <_2 a_1$ let $b \in |\mathfrak{M}_2|$ be such that $b <_2 b_1$;
2. if $a_n <_1 a$ let $b \in |\mathfrak{M}_2|$ be such that $b_n <_2 b$;
3. if $a_i <_1 a <_1 a_{i+1}$ for some i, then let $b \in |\mathfrak{M}_2|$ be such that $b_i <_2 b <_2 b_{i+1}$.

It is always possible to find a b with the desired property since $\mathfrak{M}_2$ is a dense linear ordering without endpoints. Define $q = p \cup \{\langle a, b\rangle\}$ so that $q \in \mathcal{I}$ is the desired extension of p. This establishes the Forth property. The Back property is similar. So $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$; by Theorem 1.16, $\mathfrak{M}_1 \simeq \mathfrak{M}_2$. □

Problem 1.5. Complete the proof of Theorem 1.26 by verifying that $\mathcal{I}$ satisfies the Back property.

Remark 4. Let $\mathfrak{S}$ be any enumerable dense linear ordering without endpoints. Then (by Theorem 1.26) $\mathfrak{S} \simeq \mathfrak{Q}$, where $\mathfrak{Q} = (\mathbb{Q}, <)$ is the enumerable dense linear ordering having the set $\mathbb{Q}$ of the rational numbers as its domain. Now consider again the structure $\mathfrak{R} = (\mathbb{R}, <)$ from Remark 2. We saw that there is an enumerable structure $\mathfrak{S}$ such that $\mathfrak{R} \equiv \mathfrak{S}$. But $\mathfrak{S}$ is an enumerable dense linear ordering without endpoints, and so it is isomorphic (and hence elementarily equivalent) to the structure $\mathfrak{Q}$. By transitivity of elementary equivalence, $\mathfrak{R} \equiv \mathfrak{Q}$. (We could have shown this directly by establishing $\mathfrak{R} \simeq_p \mathfrak{Q}$ by the same back-and-forth argument.)

Chapter 2

Models of Arithmetic

2.1 Introduction

The *standard model* of arithmetic is the structure $\mathfrak{N}$ with $|\mathfrak{N}| = \mathbb{N}$ in which o, $\prime$, $+$, $\times$, and $<$ are interpreted as you would expect. That is, o is 0, $\prime$ is the successor function, $+$ is interpreted as addition and $\times$ as multiplication of the numbers in $\mathbb{N}$. Specifically,

$$\begin{aligned} \mathsf{o}^{\mathfrak{N}} &= 0 \\ \prime^{\mathfrak{N}}(n) &= n + 1 \\ +^{\mathfrak{N}}(n, m) &= n + m \\ \times^{\mathfrak{N}}(n, m) &= nm \end{aligned}$$

Of course, there are structures for $\mathcal{L}_A$ that have domains other than $\mathbb{N}$. For instance, we can take $\mathfrak{M}$ with domain $|\mathfrak{M}| = \{a\}^*$ (the finite sequences of the single symbol a, i.e., $\emptyset$, a, aa, aaa, ...), and interpretations

$$\begin{aligned} \mathsf{o}^{\mathfrak{M}} &= \emptyset \\ \prime^{\mathfrak{M}}(s) &= s \frown a \\ +^{\mathfrak{M}}(n, m) &= a^{n+m} \\ \times^{\mathfrak{M}}(n, m) &= a^{nm} \end{aligned}$$

These two structures are "essentially the same" in the sense that the only difference is the elements of the domains but not how the elements of the domains are related among each other by the interpretation functions. We say that the two structures are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in $\mathfrak{N}$ also has models that are not isomorphic to $\mathfrak{N}$. Such structures are called *non-standard*. The interesting thing about them is that while the elements of a standard model (i.e., $\mathfrak{N}$, but also all structures isomorphic to it) are exhausted by the values of the standard numerals $\overline{n}$, i.e.,

$$|\mathfrak{N}| = \{\mathrm{Val}^{\mathfrak{N}}(\overline{n}) : n \in \mathbb{N}\}$$

that isn't the case in non-standard models: if $\mathfrak{M}$ is non-standard, then there is at least one $x \in |\mathfrak{M}|$ such that $x \neq \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all n.

These non-standard elements are pretty neat: they are "infinite natural numbers." But their existence also explains, in a sense, the incompleteness phenomena. Consider an example, e.g., the consistency statement for Peano arithmetic, $\mathsf{Con}_{\mathbf{PA}}$, i.e., $\lnot\exists x\, \mathsf{Prf}_{\mathbf{PA}}(x, \ulcorner\bot\urcorner)$. Since $\mathbf{PA}$ neither proves $\mathsf{Con}_{\mathbf{PA}}$ nor $\lnot\mathsf{Con}_{\mathbf{PA}}$, either can be consistently added to $\mathbf{PA}$. Since $\mathbf{PA}$ is consistent, $\mathfrak{N} \vDash \mathsf{Con}_{\mathbf{PA}}$, and consequently $\mathfrak{N} \nvDash \lnot\mathsf{Con}_{\mathbf{PA}}$. So $\mathfrak{N}$ is *not* a model of $\mathbf{PA} \cup \{\lnot\mathsf{Con}_{\mathbf{PA}}\}$, and all its models must be nonstandard. Models of $\mathbf{PA} \cup \{\lnot\mathsf{Con}_{\mathbf{PA}}\}$ must contain some element that serves as the witness that makes $\exists x\, \mathsf{Prf}_{\mathbf{PA}}(\ulcorner\bot\urcorner)$ true, i.e., a Gödel number of a derivation of a contradiction from $\mathbf{PA}$. Such an element can't be standard—since $\mathbf{PA} \vdash \lnot\mathsf{Prf}_{\mathbf{PA}}(\overline{n}, \ulcorner\bot\urcorner)$ for every n.

2.2 Standard Models of Arithmetic

mod:mar:stm: sec

The language of arithmetic $\mathcal{L}_A$ is obviously intended to be about numbers, specifically, about natural numbers. So, "the" standard model $\mathfrak{N}$ is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two structures that are isomorphic share those. So we can be a bit more liberal, and consider any structure that is isomorphic to $\mathfrak{N}$ "standard."

Definition 2.1. A structure for $\mathcal{L}_A$ is *standard* if it is isomorphic to $\mathfrak{N}$.

mod:mar:stm: prop:standard-domain

Proposition 2.2. *If a structure $\mathfrak{M}$ is standard, then its domain is the set of values of the standard numerals, i.e.,*

$$|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$$

Proof. Clearly, every $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \in |\mathfrak{M}|$. We just have to show that every $x \in |\mathfrak{M}|$ is equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some n. Since $\mathfrak{M}$ is standard, it is isomorphic to $\mathfrak{N}$. Suppose $g\colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism. Then $g(n) = g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. But for every $x \in |\mathfrak{M}|$, there is an $n \in \mathbb{N}$ such that $g(n) = x$, since g is surjective. □

If a structure $\mathfrak{M}$ for $\mathcal{L}_A$ is standard, the elements of its domain can all be named by the standard numerals $\overline{0}$, $\overline{1}$, $\overline{2}$, …, i.e., the terms o, o', o'', etc. Of course, this does not mean that the elements of $|\mathfrak{M}|$ *are* the numbers, just that we can pick them out the same way we can pick out the numbers in $|\mathfrak{N}|$. explanation

Problem 2.1. Show that the converse of Proposition 2.2 is false, i.e., give an example of a structure $\mathfrak{M}$ with $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$ that is not isomorphic to $\mathfrak{N}$.

mod:mar:stm: prop:thq-standard

Proposition 2.3. *If $\mathfrak{M} \vDash \mathbf{Q}$, and $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$, then $\mathfrak{M}$ is standard.*

Proof. We have to show that $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$. Consider the function $g\colon \mathbb{N} \to |\mathfrak{M}|$ defined by $g(n) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. By the hypothesis, g is surjective. It is also injective: $\mathbf{Q} \vdash \overline{n} \neq \overline{m}$ whenever $n \neq m$. Thus, since $\mathfrak{M} \vDash \mathbf{Q}$, $\mathfrak{M} \vDash \overline{n} \neq \overline{m}$, whenever $n \neq m$. Thus, if $n \neq m$, then $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \neq \mathrm{Val}^{\mathfrak{M}}(\overline{m})$, i.e., $g(n) \neq g(m)$.

We also have to verify that g is an isomorphism.

1. We have $g(\mathsf{o}^{\mathfrak{N}}) = g(0)$ since, $\mathsf{o}^{\mathfrak{N}} = 0$. By definition of g, $g(0) = \mathrm{Val}^{\mathfrak{M}}(\overline{0})$. But $\overline{0}$ is just o, and the value of a term which happens to be a constant symbol is given by what the structure assigns to that constant symbol, i.e., $\mathrm{Val}^{\mathfrak{M}}(\mathsf{o}) = \mathsf{o}^{\mathfrak{M}}$. So we have $g(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$ as required.

2. $g({\prime}^{\mathfrak{N}}(n)) = g(n+1)$, since $\prime$ in $\mathfrak{N}$ is the successor function on $\mathbb{N}$. Then, $g(n+1) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+1})$ by definition of g. But $\overline{n+1}$ is the same term as $\overline{n}'$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n}')$. By the definition of the value function, this is $= {\prime}^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ we get $g({\prime}^{\mathfrak{N}}(n)) = {\prime}^{\mathfrak{M}}(g(n))$.

3. $g(+^{\mathfrak{N}}(n,m)) = g(n+m)$, since $+$ in $\mathfrak{N}$ is the addition function on $\mathbb{N}$. Then, $g(n+m) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+m})$ by definition of g. But $\mathbf{Q} \vdash \overline{n+m} = (\overline{n} + \overline{m})$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+m}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n} + \overline{m})$. By the definition of the value function, this is $= +^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ and $\mathrm{Val}^{\mathfrak{M}}(\overline{m}) = g(m)$, we get $g(+^{\mathfrak{N}}(n,m)) = +^{\mathfrak{M}}(g(n), g(m))$.

4. $g(\times^{\mathfrak{N}}(n,m)) = \times^{\mathfrak{M}}(g(n), g(m))$: Exercise.

5. $\langle n, m \rangle \in <^{\mathfrak{N}}$ iff $n < m$. If $n < m$, then $\mathbf{Q} \vdash \overline{n} < \overline{m}$, and also $\mathfrak{M} \vDash \overline{n} < \overline{m}$. Thus $\langle \mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m}) \rangle \in <^{\mathfrak{M}}$, i.e., $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$. If $n \not< m$, then $\mathbf{Q} \vdash \neg\overline{n} < \overline{m}$, and consequently $\mathfrak{M} \nvDash \overline{n} < \overline{m}$. Thus, as before, $\langle g(n), g(m) \rangle \notin <^{\mathfrak{M}}$. Together, we get: $\langle n, m \rangle \in <^{\mathfrak{N}}$ iff $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$. □

explanation The function g is the most obvious way of defining a mapping from $\mathbb{N}$ to the domain of any other structure $\mathfrak{M}$ for $\mathcal{L}_A$, since every such $\mathfrak{M}$ contains elements named by $\overline{0}$, $\overline{1}$, $\overline{2}$, etc. So it isn't surprising that if $\mathfrak{M}$ makes at least some basic statements about the $\overline{n}$'s true in the same way that $\mathfrak{N}$ does, and g is also bijective, then g will turn into an isomorphism. In fact, if $|\mathfrak{M}|$ contains no elements other than what the $\overline{n}$'s name, it's the only one.

Proposition 2.4. *If $\mathfrak{M}$ is standard, then g from the proof of Proposition 2.3 is the only isomorphism from $\mathfrak{N}$ to $\mathfrak{M}$.* mod:mar:stm: prop:thq-unique-iso

Proof. Suppose $h\colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism between $\mathfrak{N}$ and $\mathfrak{M}$. We show that $g = h$ by induction on n. If $n = 0$, then $g(0) = \mathsf{o}^{\mathfrak{M}}$ by definition of g. But since h is an isomorphism, $h(0) = h(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$, so $g(0) = h(0)$.

Now consider the case for $n+1$. We have

$$\begin{aligned} g(n+1) &= \mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) \text{ by definition of } g \\ &= \mathrm{Val}^{\mathfrak{M}}(\overline{n}') \text{ since } \overline{n+1} \equiv \overline{n}' \\ &= \prime^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n})) \text{ by definition of } \mathrm{Val}^{\mathfrak{M}}(t') \\ &= \prime^{\mathfrak{M}}(g(n)) \text{ by definition of } g \\ &= \prime^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis} \\ &= h(\prime^{\mathfrak{N}}(n)) \text{ since } h \text{ is an isomorphism} \\ &= h(n+1) \end{aligned}$$

□

For any denumerable set M, there's a bijection between $\mathbb{N}$ and M, so every such set M is potentially the domain of a standard model $\mathfrak{M}$. In fact, once you pick an object $z \in M$ and a suitable function s as $\mathsf{o}^{\mathfrak{M}}$ and $\prime^{\mathfrak{M}}$, the interpretations of $+$, $\times$, and $<$ is already fixed. Only functions $s\colon M \to M \setminus \{z\}$ that are both injective and surjective are suitable in a standard model as $\prime^{\mathfrak{M}}$. The range of s cannot contain z, since otherwise $\forall x\, \mathsf{o} \neq x'$ would be false. That sentence is true in $\mathfrak{N}$, and so $\mathfrak{M}$ also has to make it true. The function s has to be injective, since the successor function $\prime^{\mathfrak{N}}$ in $\mathfrak{N}$ is, and that $\prime^{\mathfrak{N}}$ is injective is expressed by a sentence true in $\mathfrak{N}$. It has to be surjective because otherwise there would be some $x \in M \setminus \{z\}$ not in the domain of s, i.e., the sentence $\forall x\,(x = \mathsf{o} \lor \exists y\, y' = x)$ would be false in $\mathfrak{M}$—but it is true in $\mathfrak{N}$. explanation

2.3 Non-Standard Models

We call a structure for $\mathcal{L}_A$ standard if it is isomorphic to $\mathfrak{N}$. If a structure isn't isomorphic to $\mathfrak{N}$, it is called non-standard. explanation

Definition 2.5. A structure $\mathfrak{M}$ for $\mathcal{L}_A$ is *non-standard* if it is not isomorphic to $\mathfrak{N}$. The elements $x \in |\mathfrak{M}|$ which are equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some $n \in \mathbb{N}$ are called *standard numbers* (of $\mathfrak{M}$), and those not, *non-standard numbers*.

By Proposition 2.2, any standard structure for $\mathcal{L}_A$ contains only standard elements. Consequently, a non-standard structure must contain at least one non-standard element. In fact, the existence of a non-standard element guarantees that the structure is non-standard. explanation

Proposition 2.6. *If a structure $\mathfrak{M}$ for $\mathcal{L}_A$ contains a non-standard number, $\mathfrak{M}$ is non-standard.*

Proof. Suppose not, i.e., suppose $\mathfrak{M}$ standard but contains a non-standard number x. Let $g\colon \mathbb{N} \to |\mathfrak{M}|$ be an isomorphism. It is easy to see (by induction on n) that $g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. In other words, g maps standard numbers of $\mathfrak{N}$ to standard numbers of $\mathfrak{M}$. If $\mathfrak{M}$ contains a non-standard number, g cannot be surjective, contrary to hypothesis. □

Problem 2.2. Recall that $\mathbf{Q}$ contains the axioms

$$\forall x\,\forall y\,(x' = y' \rightarrow x = y) \tag{Q_1}$$
$$\forall x\,\mathsf{o} \neq x' \tag{Q_2}$$
$$\forall x\,(x = \mathsf{o} \lor \exists y\,x = y') \tag{Q_3}$$

Give structures $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ such that

1. $\mathfrak{M}_1 \vDash Q_1$, $\mathfrak{M}_1 \vDash Q_2$, $\mathfrak{M}_1 \nvDash Q_3$;
2. $\mathfrak{M}_2 \vDash Q_1$, $\mathfrak{M}_2 \nvDash Q_2$, $\mathfrak{M}_2 \vDash Q_3$; and
3. $\mathfrak{M}_3 \nvDash Q_1$, $\mathfrak{M}_3 \vDash Q_2$, $\mathfrak{M}_3 \vDash Q_3$;

Obviously, you just have to specify $\mathsf{o}^{\mathfrak{M}_i}$ and $\prime^{\mathfrak{M}_i}$ for each.

explanation

It is easy enough to specify non-standard structures for $\mathcal{L}_A$. For instance, take the structure with domain $\mathbb{Z}$ and interpret all non-logical symbols as usual. Since negative numbers are not values of $\overline{n}$ for any n, this structure is non-standard. Of course, it will not be a *model* of arithmetic in the sense that it makes the same sentences true as $\mathfrak{N}$. For instance, $\forall x\,x' \neq \mathsf{o}$ is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem.

Proposition 2.7. *Let* $\mathbf{TA} = \{\varphi : \mathfrak{N} \vDash \varphi\}$ *be the theory of* $\mathfrak{N}$. $\mathbf{TA}$ *has an enumerable non-standard model.*

Proof. Expand $\mathcal{L}_A$ by a new constant symbol c and consider the set of sentences

$$\Gamma = \mathbf{TA} \cup \{c \neq \overline{0}, c \neq \overline{1}, c \neq \overline{2}, \dots\}$$

Any model $\mathfrak{M}^c$ of Γ would contain an element $x = c^{\mathfrak{M}}$ which is non-standard, since $x \neq \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all $n \in \mathbb{N}$. Also, obviously, $\mathfrak{M}^c \vDash \mathbf{TA}$, since $\mathbf{TA} \subseteq \Gamma$. If we turn $\mathfrak{M}^c$ into a structure $\mathfrak{M}$ for $\mathcal{L}_A$ simply by forgetting about c, its domain still contains the non-standard x, and also $\mathfrak{M} \vDash \mathbf{TA}$. The latter is guaranteed since c does not occur in $\mathbf{TA}$. So, it suffices to show that Γ has a model.

We use the compactness theorem to show that Γ has a model. If every finite subset of Γ is satisfiable, so is Γ. Consider any finite subset $\Gamma_0 \subseteq \Gamma$. Γ_0 includes some sentences of $\mathbf{TA}$ and some of the form $c \neq \overline{n}$, but only finitely many. Suppose k is the largest number so that $c \neq \overline{k} \in \Gamma_0$. Define $\mathfrak{N}_k$ by expanding $\mathfrak{N}$ to include the interpretation $c^{\mathfrak{N}_k} = k + 1$. $\mathfrak{N}_k \vDash \Gamma_0$: if $\varphi \in \mathbf{TA}$, $\mathfrak{N}_k \vDash \varphi$ since $\mathfrak{N}_k$ is just like $\mathfrak{N}$ in all respects except c, and c does not occur in φ. And $\mathfrak{N}_k \vDash c \neq \overline{n}$, since $n \leq k$, and $\mathrm{Val}^{\mathfrak{N}_k}(c) = k + 1$. Thus, every finite subset of Γ is satisfiable. □

2.4 Models of Q

We know that there are non-standard structures that make the same sentences true as $\mathfrak{N}$ does, i.e., is a model of **TA**. Since $\mathfrak{N} \vDash \mathbf{Q}$, any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g., $\mathbf{Q} \nvDash \forall x\, \forall y\, (x + y) = (y + x)$. Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make $\forall x\, \forall y\, (x + y) = (y + x)$ false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly. explanation

mod:mar:mdq: ex:model-K-of-Q **Example 2.8.** Consider the structure $\mathfrak{K}$ with domain $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$$\begin{aligned}
\mathsf{o}^{\mathfrak{K}} &= 0 \\
{\prime}^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\
+^{\mathfrak{K}}(x,y) &= \begin{cases} x+y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
\times^{\mathfrak{K}}(x,y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases} \\
<^{\mathfrak{K}} &= \{\langle x, y\rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a\rangle : x \in |\mathfrak{K}|\}
\end{aligned}$$

To show that $\mathfrak{K} \vDash \mathbf{Q}$ we have to verify that all axioms of **Q** are true in $\mathfrak{K}$. For convenience, let's write x^* for ${\prime}^{\mathfrak{K}}(x)$ (the "successor" of x in $\mathfrak{K}$), $x \oplus y$ for $+^{\mathfrak{K}}(x, y)$ (the "sum" of x and y in $\mathfrak{K}$, $x \otimes y$ for $\times^{\mathfrak{K}}(x, y)$ (the "product" of x and y in $\mathfrak{K}$), and $x \lessdot y$ for $\langle x, y\rangle \in <^{\mathfrak{K}}$. With these abbreviations, we can give the operations in $\mathfrak{K}$ more perspicuously as

x	x^*
n	$n+1$
a	a

$x \oplus y$	0	m	a
0	0	m	a
n	n	$n+m$	a
a	a	a	a

$x \otimes y$	0	m	a
0	0	0	0
n	0	nm	a
a	0	a	a

We have $n \lessdot m$ iff $n < m$ for $n, m \in \mathbb{N}$ and $x \lessdot a$ for all $x \in |\mathfrak{K}|$.

$\mathfrak{K} \vDash \forall x\, \forall y\, (x' = y' \rightarrow x = y)$ since $*$ is injective. $\mathfrak{K} \vDash \forall x\, \mathsf{o} \neq x'$ since 0 is not a $*$-successor in $\mathfrak{K}$. $\mathfrak{K} \vDash \forall x\, (x = \mathsf{o} \lor \exists y\, x = y')$ since for every $n > 0$, $n = (n-1)^*$, and $a = a^*$.

$\mathfrak{K} \vDash \forall x\, (x + \mathsf{o}) = x$ since $n \oplus 0 = n + 0 = n$, and $a \oplus 0 = a$ by definition of $\oplus$. $\mathfrak{K} \vDash \forall x\, \forall y\, (x + y') = (x + y)'$ is a bit trickier. If n, m are both standard, we have:

$$(n \oplus m^*) = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*$$

since $\oplus$ and * agree with $+$ and $\prime$ on standard numbers. Now suppose $x \in |\mathfrak{K}|$. Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if $y \in |\mathfrak{K}|$ but $x = a$. Here we also have to distinguish cases according to whether $y = n$ is standard or $y = b$:

$$\begin{aligned}(a \oplus n^*) &= (a \oplus (n+1)) = a = a^* = (a \oplus n)^* \\ (a \oplus a^*) &= (a \oplus a) = a = a^* = (a \oplus a)^*\end{aligned}$$

This is of course a bit more detailed than needed. For instance, since $a \oplus z = a$ whatever z is, we can immediately conclude $a \oplus a^* = a$. The remaining axioms can be verified the same way.

$\mathfrak{K}$ is thus a model of $\mathbf{Q}$. Its "addition" $\oplus$ is also commutative. But there are other sentences true in $\mathfrak{N}$ but false in $\mathfrak{K}$, and vice versa. For instance, $a \lessdot a$, so $\mathfrak{K} \vDash \exists x\, x < x$ and $\mathfrak{K} \nvDash \forall x\, \neg x < x$. This shows that $\mathbf{Q} \nvdash \forall x\, \neg x < x$.

Problem 2.3. Prove that $\mathfrak{K}$ from Example 2.8 satisfies the remaining axioms of $\mathbf{Q}$,

$$\forall x\, (x \times 0) = 0 \tag{Q_6}$$

$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$

$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y) \tag{Q_8}$$

Find a sentence only involving $\prime$ true in $\mathfrak{N}$ but false in $\mathfrak{K}$.

Example 2.9. Consider the structure $\mathfrak{L}$ with domain $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$ and interpretations $\prime^{\mathfrak{L}} = *$, $+^{\mathfrak{L}} = \oplus$ given by mod:mar:mdq: ex:model-L-of-Q

x	x^*
n	$n+1$
a	a
b	b

$x \oplus y$	m	a	b
n	$n+m$	b	a
a	a	b	a
b	b	b	a

Since $*$ is injective, 0 is not in its range, and every $x \in |\mathfrak{L}|$ other than 0 is, axioms Q_1–Q_3 are true in $\mathfrak{L}$. For any x, $x \oplus 0 = x$, so Q_4 is true as well. For Q_5, consider $x \oplus y^*$ and $(x \oplus y)^*$. They are equal if x and y are both standard, since then $*$ and $\oplus$ agree with $\prime$ and $+$. If x is non-standard, and y is standard, we have $x \oplus y^* = x = x^* = (x \oplus y)^*$. If x and y are both non-standard, we have four cases:

$$\begin{aligned}a \oplus a^* &= b = b^* = (a \oplus a)^* \\ b \oplus b^* &= a = a^* = (b \oplus b)^* \\ b \oplus a^* &= b = b^* = (b \oplus y)^* \\ a \oplus b^* &= a = a^* = (a \oplus b)^*\end{aligned}$$

If x is standard, but y is non-standard, we have

$$n \oplus a^* = n \oplus a = b = b^* = (n \oplus a)^*$$
$$n \oplus b^* = n \oplus b = a = a^* = (n \oplus b)^*$$

So, $\mathfrak{L} \vDash Q_5$. However, $a \oplus 0 \neq 0 \oplus a$, so $\mathfrak{L} \nvDash \forall x\, \forall y\, (x + y) = (y + x)$.

Problem 2.4. Expand $\mathfrak{L}$ of Example 2.9 to include $\otimes$ and $\lessdot$ that interpret $\times$ and $<$. Show that your structure satisfies the remaining axioms of **Q**,

$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$
$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$
$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y) \tag{Q_8}$$

Problem 2.5. In $\mathfrak{L}$ of Example 2.9, $a^* = a$ and $b^* = b$. Is there a model of **Q** in which $a^* = b$ and $b^* = a$?

We've explicitly constructed models of **Q** in which the non-standard elements live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard element x cannot be $\lessdot 0$, since $\mathbf{Q} \vdash \forall x\, \neg x < 0$ (see **??**). Also, for every n, $\mathbf{Q} \vdash \forall x\, (x < \overline{n}' \rightarrow (x = \overline{0} \vee x = \overline{1} \vee \dots \vee x = \overline{n}))$ (**??**), so we can't have $a \lessdot n$ for any $n > 0$. explanation

2.5 Models of PA

Any non-standard model of **TA** is also one of **PA**. We know that non-standard models of **TA** and hence of **PA** exist. We also know that such non-standard models contain non-standard "numbers," i.e., elements of the domain that are "beyond" all the standard "numbers." But how are they arranged? How many are there? We've seen that models of the weaker theory **Q** can contain as few as a single non-standard number. But these simple structures are not models of **PA** or **TA**. explanation

The key to understanding the structure of models of **PA** or **TA** is to see what facts are derivable in these theories. For instance, already **PA** proves that $\forall x\, x \neq x'$ and $\forall x\, \forall y\, (x + y) = (y + x)$, so this rules out simple structures (in which these sentences are false) as models of **PA**.

Suppose $\mathfrak{M}$ is a model of **PA**. Then if $\mathbf{PA} \vdash \varphi$, $\mathfrak{M} \vDash \varphi$. Let's again use $\mathbf{z}$ for $\mathsf{o}^{\mathfrak{M}}$, $*$ for $'^{\mathfrak{M}}$, $\oplus$ for $+^{\mathfrak{M}}$, $\otimes$ for $\times^{\mathfrak{M}}$, and $\lessdot$ for $<^{\mathfrak{M}}$. Any sentence φ then states some condition about $\mathbf{z}$, $*$, $\oplus$, $\otimes$, and $\lessdot$, and if $\mathfrak{M} \vDash \varphi$ that condition must be satisfied. For instance, if $\mathfrak{M} \vDash Q_1$, i.e., $\mathfrak{M} \vDash \forall x\, \forall y\, (x' = y' \rightarrow x = y)$, then $*$ must be injective.

Proposition 2.10. *In $\mathfrak{M}$, $\lessdot$ is a linear strict order, i.e., it satisfies:*

1. *Not $x \lessdot x$ for any $x \in |\mathfrak{M}|$.*
2. *If $x \lessdot y$ and $y \lessdot z$ then $x \lessdot z$.*

3. *For any $x \neq y$, $x \lessdot y$ or $y \lessdot x$*

Proof. **PA** proves:

1. $\forall x\, \neg x < x$
2. $\forall x\, \forall y\, \forall z\, ((x < y \land y < z) \to x < z)$
3. $\forall x\, \forall y\, ((x < y \lor y < x) \lor x = y))$ $\square$

Proposition 2.11. *$\mathbf{z}$ is the least element of $|\mathfrak{M}|$ in the $\lessdot$-ordering. For any x, $x \lessdot x^*$, and x^* is the $\lessdot$-least element with that property. For any x, there is a unique y such that $y^* = x$. (We call y the "predecessor" of x in $\mathfrak{M}$, and denote it by ${}^*x$.)* mod:mar:mpa: prop:M-discrete

Proof. Exercise. $\square$

Problem 2.6. Find sentences in $\mathcal{L}_A$ derivable in **PA** (and hence true in $\mathfrak{N}$) which guarantee the properties of $\mathbf{z}$, $*$, and $\lessdot$ in Proposition 2.11

Proposition 2.12. *All standard elements of $\mathfrak{M}$ are less than (according to $\lessdot$) all non-standard elements.*

Proof. We'll use n as short for $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$, a standard element of $\mathfrak{M}$. Already **Q** proves that, for any $n \in \mathbb{N}$, $\forall x\, (x < \overline{n}' \to (x = \overline{0} \lor x = \overline{1} \lor \cdots \lor x = \overline{n}))$. There are no elements that are $\lessdot \mathbf{z}$. So if n is standard and x is non-standard, we cannot have $x \lessdot n$. By definition, a non-standard element is one that isn't $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for any $n \in \mathbb{N}$, so $x \neq n$ as well. Since $\lessdot$ is a linear order, we must have $n \lessdot x$. $\square$

Proposition 2.13. *Every nonstandard element x of $|\mathfrak{M}|$ is an element of the subset*

$$\ldots{}^{***}x \lessdot {}^{**}x \lessdot {}^{*}x \lessdot x \lessdot x^* \lessdot x^{**} \lessdot x^{***} \lessdot \ldots$$

We call this subset the block of x *and write it as $[x]$. It has no least and no greatest element. It can be characterized as the set of those $y \in |\mathfrak{M}|$ such that, for some standard n, $x \oplus n = y$ or $y \oplus n = x$.*

Proof. Clearly, such a set $[x]$ always exists since every element y of $|\mathfrak{M}|$ has a unique successor y^* and unique predecessor ${}^*y$. For successive elements y, y^* we have $y \lessdot y^*$ and y^* is the $\lessdot$-least element of $|\mathfrak{M}|$ such that y is $\lessdot$-less than it. Since always ${}^*y \lessdot y$ and $y \lessdot y^*$, $[x]$ has no least or greatest element. If $y \in [x]$ then $x \in [y]$, for then either $y^{*\ldots*} = x$ or $x^{*\ldots*} = y$. If $y^{*\ldots*} = x$ (with n $*$'s), then $y \oplus n = x$ and conversely, since $\mathbf{PA} \vdash \forall x\, x'^{\ldots\prime} = (x + \overline{n})$ (if n is the number of $\prime$'s). $\square$

Proposition 2.14. *If $[x] \neq [y]$ and $x \lessdot y$, then for any $u \in [x]$ and any $v \in [y]$, $u \lessdot v$.*

Proof. Note that $\mathbf{PA} \vdash \forall x \, \forall y \, (x < y \to (x' < y \lor x' = y))$. Thus, if $u \lessdot v$, we also have $u \oplus n^* \lessdot v$ for any n if $[u] \neq [v]$.

Any $u \in [x]$ is $\lessdot y$: $x \lessdot y$ by assumption. If $u \lessdot x$, $u \lessdot y$ by transitivity. And if $x \lessdot u$ but $u \in [x]$, we have $u = x \oplus n^*$ for some n, and so $u \lessdot y$ by the fact just proved.

Now suppose that $v \in [y]$ is $\lessdot y$, i.e., $v \oplus m^* = y$ for some standard m. This rules out $v \lessdot x$, otherwise $y = v \oplus m^* \lessdot x$. Clearly also, $x \neq v$, otherwise $x \oplus m^* = v \oplus m^* = y$ and we would have $[x] = [y]$. So, $x \lessdot v$. But then also $x \oplus n^* \lessdot v$ for any n. Hence, if $x \lessdot u$ and $u \in [x]$, we have $u \lessdot v$. If $u \lessdot x$ then $u \lessdot v$ by transitivity.

Lastly, if $y \lessdot v$, $u \lessdot v$ since, as we've shown, $u \lessdot y$ and $y \lessdot v$. □

Corollary 2.15. *If $[x] \neq [y]$, $[x] \cap [y] = \emptyset$.*

Proof. Suppose $z \in [x]$ and $x \lessdot y$. Then $z \lessdot u$ for all $u \in [y]$. If $z \in [y]$, we would have $z \lessdot z$. Similarly if $y \lessdot x$. □

This means that the blocks themselves can be ordered in a way that respects $\lessdot$: $[x] \lessdot [y]$ iff $x \lessdot y$, or, equivalently, if $u \lessdot v$ for any $u \in [x]$ and $v \in [y]$. Clearly, the standard block $[0]$ is the least block. It intersects with no non-standard block, and no two non-standard blocks intersect either. Specifically, you cannot "reach" a different block by taking repeated successors or predecessors. explanation

Proposition 2.16. *If x and y are non-standard, then $x \lessdot x \oplus y$ and $x \oplus y \notin [x]$.*

Proof. If y is nonstandard, then $y \neq \mathbf{z}$. $\mathbf{PA} \vdash \forall x \, (y \neq \mathsf{o} \to x < (x + y))$. Now suppose $x \oplus y \in [x]$. Since $x \lessdot x \oplus y$, we would have $x \oplus n^* = x \oplus y$. But $\mathbf{PA} \vdash \forall x \, \forall y \, \forall z \, ((x + y) = (x + z) \to y = z)$ (the cancellation law for addition). This would mean $y = n^*$ for some standard n; but y is assumed to be non-standard. □

Proposition 2.17. *There is no least non-standard block.*

Proof. $\mathbf{PA} \vdash \forall x \, \exists y \, ((y + y) = x \lor (y + y)' = x)$, i.e., that every x is divisible by 2 (possibly with remainder 1). If x is non-standard, so is y. By the preceding proposition, $y \lessdot y \oplus y$ and $y \oplus y \notin [y]$. Then also $y \lessdot (y \oplus y)^*$ and $(y \oplus y)^* \notin [y]$. But $x = y \oplus y$ or $x = (y \oplus y)^*$, so $y \lessdot x$ and $y \notin [x]$. □

Proposition 2.18. *There is no largest block.*

Proof. Exercise. □

Problem 2.7. Show that in a non-standard model of $\mathbf{PA}$, there is no largest block.

mod:mar:mpa: prop:blocks-dense **Proposition 2.19.** *The ordering of the blocks is dense. That is, if $x \lessdot y$ and $[x] \neq [y]$, then there is a block $[z]$ distinct from both that is between them.*

Proof. Suppose $x \lessdot y$. As before, $x \oplus y$ is divisible by two (possibly with remainder): there is a $z \in |\mathfrak{M}|$ such that either $x \oplus y = z \oplus z$ or $x \oplus y = (z \oplus z)^*$. The element z is the "average" of x and y, and $x \lessdot z$ and $z \lessdot y$. □

Problem 2.8. Write out a detailed proof of Proposition 2.19. Which sentence must **PA** derive in order to guarantee the existence of z? Why is $x \lessdot z$ and $z \lessdot y$, and why is $[x] \neq [z]$ and $[z] \neq [y]$?

explanation The non-standard blocks are therefore ordered like the rationals: they form a denumerable dense linear ordering without endpoints. One can show that any two such denumerable orderings are isomorphic. It follows that for any two enumerable non-standard models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of true arithmetic, their reducts to the language containing $<$ and $=$ only are isomorphic. Indeed, an isomorphism h can be defined as follows: the standard parts of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic to the standard model $\mathfrak{N}$ and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore isomorphic; an isomorphism of the blocks can be extended to an isomorphism *within* the blocks by matching up arbitrary elements in each, and then taking the image of the successor of x in $\mathfrak{M}_1$ to be the successor of the image of x in $\mathfrak{M}_2$. Note that it does *not* follow that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic in the full language of arithmetic (indeed, isomorphism is always relative to a language), as there are non-isomorphic ways to define addition and multiplication over $|\mathfrak{M}_1|$ and $|\mathfrak{M}_2|$. (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be 2.)

2.6 Computable Models of Arithmetic

explanation The standard model $\mathfrak{N}$ has two nice features. Its domain is the natural numbers $\mathbb{N}$, i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral $\overline{n}$ actually picks out n. The other nice feature is that the interpretations of the non-logical symbols of $\mathcal{L}_A$ are all *computable*. The successor, addition, and multiplication functions which serve as $\prime^{\mathfrak{N}}$, $+^{\mathfrak{N}}$, and $\times^{\mathfrak{N}}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on $\mathfrak{N}$, i.e., $<^{\mathfrak{N}}$, is decidable.

Non-standard models of arithmetical theories such as **Q** and **PA** must contain non-standard elements. Thus their domains typically include elements in addition to $\mathbb{N}$. However, any countable structure can be built on any denumerable set, including $\mathbb{N}$. So there are also non-standard models with domain $\mathbb{N}$. In such models $\mathfrak{M}$, of course, at least some numbers cannot play the roles they usually play, since some k must be different from $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all $n \in \mathbb{N}$.

Definition 2.20. A structure $\mathfrak{M}$ for $\mathcal{L}_A$ is *computable* iff $|\mathfrak{M}| = \mathbb{N}$ and $\prime^{\mathfrak{M}}$, $+^{\mathfrak{M}}$, $\times^{\mathfrak{M}}$ are computable functions and $<^{\mathfrak{M}}$ is a decidable relation.

mod:mar:cmp: ex:comp-model-q **Example 2.21.** Recall the structure $\mathfrak{K}$ from Example 2.8. Its domain was $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations

$$\mathsf{o}^{\mathfrak{K}} = 0$$

$$\prime^{\mathfrak{K}}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases}$$

$$+^{\mathfrak{K}}(x,y) = \begin{cases} x+y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$$

$$\times^{\mathfrak{K}}(x,y) = \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases}$$

$$<^{\mathfrak{K}} = \{\langle x,y\rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x,a\rangle : n \in |\mathfrak{K}|\}$$

But $|\mathfrak{K}|$ is denumerable and so is equinumerous with $\mathbb{N}$. For instance, $g\colon \mathbb{N} \to |\mathfrak{K}|$ with $g(0) = a$ and $g(n) = n+1$ for $n > 0$ is a bijection. We can turn it into an isomorphism between a new model $\mathfrak{K}'$ of $\mathbf{Q}$ and $\mathfrak{K}$. In $\mathfrak{K}'$, we have to assign different functions and relations to the symbols of $\mathcal{L}_A$, since different elements of $\mathbb{N}$ play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of a, not of the smallest standard number. The smallest standard number is now 1. So we assign $\mathsf{o}^{\mathfrak{K}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n+1$. But 0 now plays the role of a, which is its own successor. So $\prime^{\mathfrak{K}'}(0) = 0$. For addition and multiplication we likewise have

$$+^{\mathfrak{K}'}(x,y) = \begin{cases} x+y-1 & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\times^{\mathfrak{K}'}(x,y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ xy - x - y + 2 & \text{if } x, y > 1 \\ 0 & \text{otherwise} \end{cases}$$

And we have $\langle x,y\rangle \in <^{\mathfrak{K}'}$ iff $x < y$ and $x > 0$ and $y > 0$, or if $y = 0$.

All of these functions are computable functions of natural numbers and $<^{\mathfrak{K}'}$ is a decidable relation on $\mathbb{N}$—but they are not the same functions as successor, addition, and multiplication on $\mathbb{N}$, and $<^{\mathfrak{K}'}$ is not the same relation as $<$ on $\mathbb{N}$.

Problem 2.9. Give a structure $\mathfrak{L}'$ with $|\mathfrak{L}'| = \mathbb{N}$ isomorphic to $\mathfrak{L}$ of Example 2.9.

Example 2.21 shows that $\mathbf{Q}$ has computable non-standard models with domain $\mathbb{N}$. However, the following result shows that this is not true for models of $\mathbf{PA}$ (and thus also for models of $\mathbf{TA}$). explanation

Theorem 2.22 (Tennenbaum's Theorem). *$\mathfrak{N}$ is the only computable model of* $\mathbf{PA}$*.*

Chapter 3

The Interpolation Theorem

3.1 Introduction

mod:int:int:sec

The interpolation theorem is the following result: Suppose $\vDash \varphi \to \psi$. Then there is a sentence χ such that $\vDash \varphi \to \chi$ and $\vDash \chi \to \psi$. Moreover, every constant symbol, function symbol, and predicate symbol (other than $=$) in χ occurs both in φ and ψ. The sentence χ is called an *interpolant* of φ and ψ.

The interpolation theorem is interesting in its own right, but its main importance lies in the fact that it can be used to prove results about definability in a theory, and the conditions under which combining two consistent theories results in a consistent theory. The first result is known as the Beth definability theorem; the second, Robinson's joint consistency theorem.

3.2 Separation of Sentences

mod:int:sep:sec

A bit of groundwork is needed before we can proceed with the proof of the interpolation theorem. An interpolant for φ and ψ is a sentence χ such that $\varphi \vDash \chi$ and $\chi \vDash \psi$. By contraposition, the latter is true iff $\neg\psi \vDash \neg\chi$. A sentence χ with this property is said to *separate* φ and $\neg\psi$. So finding an interpolant for φ and ψ amounts to finding a sentence that separates φ and $\neg\psi$. As so often, it will be useful to consider a generalization: a sentence that separates two *sets* of sentences.

Definition 3.1. A sentence χ *separates* sets of sentences Γ and Δ if and only if $\Gamma \vDash \chi$ and $\Delta \vDash \neg\chi$. If no such sentence exists, then Γ and Δ are *inseparable*.

The inclusion relations between the classes of models of Γ, Δ and χ are represented below:

mod:int:sep:lem:sep1

Lemma 3.2. *Suppose $\mathcal{L}_0$ is the language containing every constant symbol, function symbol and predicate symbol (other than $\doteq$) that occurs in* both *Γ and Δ, and let $\mathcal{L}'_0$ be obtained by the addition of infinitely many new constant symbols c_n for $n \geq 0$. Then if Γ and Δ are inseparable in $\mathcal{L}_0$, they are also inseparable in $\mathcal{L}'_0$.*

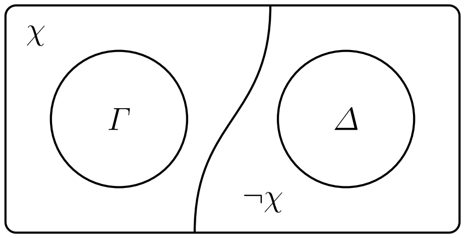

Figure 3.1: χ separates Γ and Δ

mod:int:sep: fig:sep

Proof. We proceed indirectly: suppose by way of contradiction that Γ and Δ are separated in $\mathcal{L}_0'$. Then $\Gamma \vDash \chi[c/x]$ and $\Delta \vDash \neg\chi[c/x]$ for some $\chi \in \mathcal{L}_0$ (where c is a new constant symbol—the case where χ contains more than one such new constant symbol is similar). By compactness, there are *finite* subsets Γ_0 of Γ and Δ_0 of Δ such that $\Gamma_0 \vDash \chi[c/x]$ and $\Delta_0 \vDash \neg\chi[c/x]$. Let γ be the conjunction of all formulas in Γ_0 and δ the conjunction of all formulas in Δ_0. Then

$$\gamma \vDash \chi[c/x], \qquad \delta \vDash \neg\chi[c/x].$$

From the former, by Generalization, we have $\gamma \vDash \forall x\, \chi$, and from the latter by contraposition, $\chi[c/x] \vDash \neg\delta$, whence also $\forall x\, \chi \vDash \neg\delta$. Contraposition again gives $\delta \vDash \neg\forall x\, \chi$. By monotonicity,

$$\Gamma \vDash \forall x\, \chi, \qquad \Delta \vDash \neg\forall x\, \chi,$$

so that $\forall x\, \chi$ separates Γ and Δ in $\mathcal{L}_0$. □

mod:int:sep: lem:sep2

Lemma 3.3. *Suppose that $\Gamma \cup \{\exists x\, \sigma\}$ and Δ are inseparable, and c is a new constant symbol not in Γ, Δ, or σ. Then $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ are also inseparable.*

Proof. Suppose for contradiction that χ separates $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ, while at the same time $\Gamma \cup \{\exists x \sigma\}$ and Δ are inseparable. We distinguish two cases:

1. c does not occur in χ: in this case $\Gamma \cup \{\exists x\, \sigma, \neg\chi\}$ is satisfiable (otherwise χ separates $\Gamma \cup \{\exists x\, \sigma\}$ and Δ). It remains so if $\sigma[c/x]$ is added, so χ does not separate $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ after all.

2. c does occur in χ so that χ has the form $\chi[c/x]$. Then we have that

$$\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\} \vDash \chi[c/x],$$

whence $\Gamma, \exists x\, \sigma \vDash \forall x\, (\sigma \to \chi)$ by the Deduction Theorem and Generalization, and finally $\Gamma \cup \{\exists x\, \sigma\} \vDash \exists x\, \chi$. On the other hand, $\Delta \vDash \neg\chi[c/x]$ and hence by Generalization $\Delta \vDash \neg\exists x\, \chi$. So $\Gamma \cup \{\exists x\, \sigma\}$ and Δ are separable, a contradiction. □

3.3 Craig's Interpolation Theorem

mod:int:prf: sec

Theorem 3.4 (Craig's Interpolation Theorem). *If $\vDash \varphi \rightarrow \psi$, then there is a sentence χ such that $\vDash \varphi \rightarrow \chi$ and $\vDash \chi \rightarrow \psi$, and every constant symbol, function symbol, and predicate symbol (other than $=$) in χ occurs both in φ and ψ. The sentence χ is called an* interpolant *of φ and ψ.* *mod:int:prf: thm:interpol*

Proof. Suppose $\mathcal{L}_1$ is the language of φ and $\mathcal{L}_2$ is the language of ψ. Let $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$. For each $i \in \{0, 1, 2\}$, let $\mathcal{L}_i'$ be obtained from $\mathcal{L}_i$ by adding the infinitely many new constant symbols $c_0, c_1, c_2, \dots$.

If φ is unsatisfiable, $\exists x\, x \neq x$ is an interpolant. If $\neg\psi$ is unsatisfiable (and hence ψ is valid), $\exists x\, x = x$ is an interpolant. So we may assume also that both φ and $\neg\psi$ are satisfiable.

In order to prove the contrapositive of the Interpolation Theorem, assume that there is no interpolant for φ and ψ. In other words, assume that $\{\varphi\}$ and $\{\neg\psi\}$ are inseparable in $\mathcal{L}_0$.

Our goal is to extend the pair $(\{\varphi\}, \{\neg\psi\})$ to a maximally inseparable pair (Γ^*, Δ^*). Let φ_0, φ_1, φ_2, ... enumerate the sentences of $\mathcal{L}_1$, and ψ_0, ψ_1, ψ_2, ... enumerate the sentences of $\mathcal{L}_2$. We define two increasing sequences of sets of sentences (Γ_n, Δ_n), for $n \geq 0$, as follows. Put $\Gamma_0 = \{\varphi\}$ and $\Delta_0 = \{\neg\psi\}$. Assuming (Γ_n, Δ_n) are already defined, define Γ_{n+1} and Δ_{n+1} by:

1. If $\Gamma_n \cup \{\varphi_n\}$ and Δ_n are inseparable in $\mathcal{L}_0'$, put φ_n in Γ_{n+1}. Moreover, if φ_n is an existential formula $\exists x\, \sigma$ then pick a new constant symbol c not occurring in Γ_n, Δ_n, φ_n or ψ_n, and put $\sigma[c/x]$ in Γ_{n+1}.

2. If Γ_{n+1} and $\Delta_n \cup \{\psi_n\}$ are inseparable in $\mathcal{L}_0'$, put ψ_n in Δ_{n+1}. Moreover, if ψ_n is an existential formula $\exists x\, \sigma$, then pick a new constant symbol c not occurring in Γ_{n+1}, Δ_n, φ_n or ψ_n, and put $\sigma[c/x]$ in Δ_{n+1}.

Finally, define:

$$\Gamma^* = \bigcup_{n \geq 0} \Gamma_n, \qquad \Delta^* = \bigcup_{n \geq 0} \Delta_n.$$

By simultaneous induction on n we can now prove:

1. Γ_n and Δ_n are inseparable in $\mathcal{L}_0'$; mod:int:prf: part-a

2. Γ_{n+1} and Δ_n are inseparable in $\mathcal{L}_0'$. mod:int:prf: part-b

The basis for (1) is given by Lemma 3.2. For part (2), we need to distinguish three cases:

1. If $\Gamma_0 \cup \{\varphi_0\}$ and Δ_0 are separable, then $\Gamma_1 = \Gamma_0$ and (2) is just (1);

2. If $\Gamma_1 = \Gamma_0 \cup \{\varphi_0\}$, then Γ_1 and Δ_0 are inseparable by construction.

3. It remains to consider the case where φ_0 is existential, so that $\Gamma_1 = \Gamma_0 \cup \{\exists x\, \sigma, \sigma[c/x]\}$. By construction, $\Gamma_0 \cup \{\exists x\, \sigma\}$ and Δ_0 are inseparable, so that by Lemma 3.3 also $\Gamma_0 \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ_0 are inseparable.

This completes the basis of the induction for (1) and (2) above. Now for the inductive step. For (1), if $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ then Γ_{n+1} and Δ_{n+1} are inseparable by construction (even when ψ_n is existential, by Lemma 3.3); if $\Delta_{n+1} = \Delta_n$ (because Γ_{n+1} and $\Delta_n \cup \{\psi_n\}$ are separable), then we use the induction hypothesis on (2). For the inductive step for (2), if $\Gamma_{n+2} = \Gamma_{n+1} \cup \{\varphi_{n+1}\}$ then Γ_{n+2} and Δ_{n+1} are inseparable by construction (even when φ_{n+1} is existential, by Lemma 3.3); and if $\Gamma_{n+2} = \Gamma_{n+1}$ then we use the inductive case for (1) just proved. This concludes the induction on (1) and (2).

It follows that Γ^* and Δ^* are inseparable; if not, by compactness, there is $n \geq 0$ that separates Γ_n and Δ_n, against (1). In particular, Γ^* and Δ^* are consistent: for if the former or the latter is inconsistent, then they are separated by $\exists x\, x \neq x$ or $\forall x\, x = x$, respectively.

We now show that Γ^* is maximally consistent in $\mathcal{L}'_1$ and likewise Δ^* in $\mathcal{L}'_2$. For the former, suppose that $\varphi_n \notin \Gamma^*$ and $\neg\varphi_n \notin \Gamma^*$, for some $n \geq 0$. If $\varphi_n \notin \Gamma^*$ then $\Gamma_n \cup \{\varphi_n\}$ is separable from Δ_n, and so there is $\chi \in \mathcal{L}'_0$ such that both:

$$\Gamma^* \vDash \varphi_n \to \chi, \qquad\qquad \Delta^* \vDash \neg\chi.$$

Likewise, if $\neg\varphi_n \notin \Gamma^*$, there is $\chi' \in \mathcal{L}'_0$ such that both:

$$\Gamma^* \vDash \neg\varphi_n \to \chi', \qquad\qquad \Delta^* \vDash \neg\chi'.$$

By propositional logic, $\Gamma^* \vDash \chi \lor \chi'$ and $\Delta^* \vDash \neg(\chi \lor \chi')$, so $\chi \lor \chi'$ separates $\Gamma^*$and Δ^*. A similar argument establishes that Δ^* is maximal.

Finally, we show that $\Gamma^* \cap \Delta^*$ is maximally consistent in $\mathcal{L}'_0$. It is obviously consistent, since it is the intersection of consistent sets. To show maximality, let $\sigma \in \mathcal{L}'_0$. Now, Γ^* is maximal in $\mathcal{L}'_1 \supseteq \mathcal{L}'_0$, and similarly Δ^* is maximal in $\mathcal{L}'_2 \supseteq \mathcal{L}'_0$. It follows that either $\sigma \in \Gamma^*$ or $\neg\sigma \in \Gamma^*$, and either $\sigma \in \Delta^*$ or $\neg\sigma \in \Delta^*$. If $\sigma \in \Gamma^*$ and $\neg\sigma \in \Delta^*$ then σ would separate Γ^* and Δ^*; and if $\neg\sigma \in \Gamma^*$ and $\sigma \in \Delta^*$ then Γ^* and Δ^* would be separated by $\neg\sigma$. Hence, either $\sigma \in \Gamma^* \cap \Delta^*$ or $\neg\sigma \in \Gamma^* \cap \Delta^*$, and $\Gamma^* \cap \Delta^*$ is maximal.

Since Γ^* is maximally consistent, it has a model $\mathfrak{M}'_1$ whose domain $|\mathfrak{M}'_1|$ comprises all and only the elements $c^{\mathfrak{M}'_1}$ interpreting the constant symbols—just like in the proof of the completeness theorem (**??**). Similarly, Δ^* has a model $\mathfrak{M}'_2$ whose domain $|\mathfrak{M}'_2|$ is given by the interpretations $c^{\mathfrak{M}'_2}$ of the constant symbols.

Let $\mathfrak{M}_1$ be obtained from $\mathfrak{M}'_1$ by dropping interpretations for constant symbols, function symbols, and predicate symbols in $\mathcal{L}'_1 \setminus \mathcal{L}'_0$, and similarly for $\mathfrak{M}_2$. Then the map $h\colon M_1 \to M_2$ defined by $h(c^{\mathfrak{M}'_1}) = c^{\mathfrak{M}'_2}$ is an isomorphism in $\mathcal{L}'_0$, because $\Gamma^* \cap \Delta^*$ is maximally consistent in $\mathcal{L}'_0$, as shown. This follows because any $\mathcal{L}'_0$-sentence either belongs to both Γ^* and Δ^*, or to neither: so $c^{\mathfrak{M}'_1} \in P^{\mathfrak{M}'_1}$ if and only if $P(c) \in \Gamma^*$ if and only if $P(c) \in \Delta^*$ if and only if

$c^{\mathfrak{M}'_2} \in P^{\mathfrak{M}'_2}$. The other conditions satisfied by isomorphisms can be established similarly.

Let us now define a model $\mathfrak{M}$ for the language $\mathcal{L}_1 \cup \mathcal{L}_2$ as follows:

1. The domain $|\mathfrak{M}|$ is just $|\mathfrak{M}_2|$, i.e., the set of all elements $c^{\mathfrak{M}'_2}$;
2. If a predicate symbol P is in $\mathcal{L}_2 \setminus \mathcal{L}_1$ then $P^{\mathfrak{M}} = P^{\mathfrak{M}'_2}$;
3. If a predicate P is in $\mathcal{L}_1 \setminus \mathcal{L}_2$ then $P^{\mathfrak{M}} = h(P^{\mathfrak{M}'_2})$, i.e., $\langle c_1^{\mathfrak{M}'_2}, \dots, c_n^{\mathfrak{M}'_2} \rangle \in P^{\mathfrak{M}}$ if and only if $\langle c_1^{\mathfrak{M}'_1}, \dots, c_n^{\mathfrak{M}'_1} \rangle \in P^{\mathfrak{M}'_1}$.
4. If a predicate symbol P is in $\mathcal{L}_0$ then $P^{\mathfrak{M}} = P^{\mathfrak{M}'_2} = h(P^{\mathfrak{M}'_1})$.
5. Function symbols of $\mathcal{L}_1 \cup \mathcal{L}_2$, including constant symbols, are handled similarly.

Finally, one shows by induction on formulas that $\mathfrak{M}$ agrees with $\mathfrak{M}'_1$ on all formulas of $\mathcal{L}'_1$ and with $\mathfrak{M}'_2$ on all formulas of $\mathcal{L}'_2$. In particular, $\mathfrak{M} \vDash \Gamma^* \cup \Delta^*$, whence $\mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \vDash \neg\psi$, and $\nvDash \varphi \rightarrow \psi$. This concludes the proof of Craig's Interpolation Theorem. □

3.4 The Definability Theorem

One important application of the interpolation theorem is Beth's definability theorem. To define an n-place relation R we can give a formula χ with n free variables which does not involve R. This would be an *explicit* definition of R in terms of χ. We can then say also that a theory $\Sigma(P)$ in a language containing the n-place predicate symbol P explicitly defines P if it contains (or at least entails) a formalized explicit definition, i.e., mod:int:def:sec

$$\Sigma(P) \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)).$$

But an explicit definition is only one way of defining—in the sense of determining completely—a relation. A theory may also be such that the interpretation of P is fixed by the interpretation of the rest of the language in any model. The definability theorem states that whenever a theory fixes the interpretation of P in this way—whenever it *implicitly defines* P—then it also explicitly defines it.

Definition 3.5. Suppose $\mathcal{L}$ is a language not containing the predicate symbol P. A set $\Sigma(P)$ of sentences of $\mathcal{L} \cup \{P\}$ *explicitly defines* P if and only if there is a formula $\chi(x_1, \dots, x_n)$ of $\mathcal{L}$ such that

$$\Sigma(P) \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)).$$

Definition 3.6. Suppose $\mathcal{L}$ is a language not containing the predicate symbols P and P'. A set $\Sigma(P)$ of sentences of $\mathcal{L} \cup \{P\}$ *implicitly defines* P if and only if

$$\Sigma(P) \cup \Sigma(P') \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow P'(x_1, \dots, x_n)),$$

where $\Sigma(P')$ is the result of uniformly replacing P with P' in $\Sigma(P)$.

In other words, for any model $\mathfrak{M}$ and $R, R' \subseteq |\mathfrak{M}|^n$, if both $(\mathfrak{M}, R) \vDash \Sigma(P)$ and $(\mathfrak{M}, R') \vDash \Sigma(P')$, then $R = R'$; where $(\mathfrak{M}, R)$ is the structure $\mathfrak{M}'$ for the expansion of $\mathcal{L}$ to $\mathcal{L} \cup \{P\}$ such that $P^{\mathfrak{M}'} = R$, and similarly for $(\mathfrak{M}, R')$.

Theorem 3.7 (Beth Definability Theorem). *A set $\Sigma(P)$ of $\mathcal{L} \cup \{P\}$-formulas implicitly defines P if and only $\Sigma(P)$ explicitly defines P.*

Proof. If $\Sigma(P)$ explicitly defines P then both

$$\begin{aligned} \Sigma(P) &\vDash \quad \forall x_1 \ldots \forall x_n \, (P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)) \\ \Sigma(P') &\vDash \quad \forall x_1 \ldots \forall x_n \, (P'(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)) \end{aligned}$$

and the conclusion follows. For the converse: assume that $\Sigma(P)$ implicitly defines P. First, we add constant symbols $c_1, \ldots, c_n$ to $\mathcal{L}$. Then

$$\Sigma(P) \cup \Sigma(P') \vDash P(c_1, \ldots, c_n) \to P'(c_1, \ldots, c_n).$$

By compactness, there are finite sets $\Delta_0 \subseteq \Sigma(P)$ and $\Delta_1 \subseteq \Sigma(P')$ such that

$$\Delta_0 \cup \Delta_1 \vDash P(c_1, \ldots, c_n) \to P'(c_1, \ldots, c_n).$$

Let $\theta(P)$ be the conjunction of all sentences $\varphi(P)$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$ and let $\theta(P')$ be the conjunction of all sentences $\varphi(P')$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$. Then $\theta(P) \land \theta(P') \vDash P(c_1, \ldots, c_n) \to P' c_1 \ldots c_n$. We can re-arrange this so that each predicate symbol occurs on one side of $\vDash$:

$$\theta(P) \land P(c_1, \ldots, c_n) \vDash \theta(P') \to P'(c_1, \ldots, c_n).$$

By Craig's Interpolation Theorem there is a sentence $\chi(c_1, \ldots, c_n)$ not containing P or P' such that:

$$\theta(P) \land P(c_1, \ldots, c_n) \vDash \chi(c_1, \ldots, c_n); \quad \chi(c_1, \ldots, c_n) \vDash \theta(P') \to P'(c_1, \ldots, c_n).$$

From the former of these two entailments we have: $\theta(P) \vDash P(c_1, \ldots, c_n) \to \chi(c_1, \ldots, c_n)$. And from the latter, since an $\mathcal{L} \cup \{P\}$-model $(\mathfrak{M}, R) \vDash \varphi(P)$ if and only if the corresponding $\mathcal{L} \cup \{P'\}$-model $(\mathfrak{M}, R) \models \varphi(P')$, we have $\chi(c_1, \ldots, c_n) \vDash \theta(P) \to P(c_1, \ldots, c_n)$, from which:

$$\theta(P) \vDash \chi(c_1, \ldots, c_n) \to P(c_1, \ldots, c_n).$$

Putting the two together, $\theta(P) \vDash P(c_1, \ldots, c_n) \leftrightarrow \chi(c_1, \ldots, c_n)$, and by monotonicity and generalization also

$$\Sigma(P) \vDash \forall x_1 \ldots \forall x_n \, (P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)). \qquad \square$$

Chapter 4

Lindström's Theorem

4.1 Introduction

In this chapter we aim to prove Lindström's characterization of first-order logic as the maximal logic for which (given certain further constraints) the Compactness and the Downward Löwenheim–Skolem theorems hold (**??** and **??**). First, we need a more general characterization of the general class of logics to which the theorem applies. We will restrict ourselves to *relational* languages, i.e., languages which only contain predicate symbols and individual constants, but no function symbols.

4.2 Abstract Logics

Definition 4.1. An *abstract logic* is a pair $\langle L, \models_L \rangle$, where L is a function that assigns to each language $\mathcal{L}$ a set $L(\mathcal{L})$ of sentences, and $\models_L$ is a relation between structures for the language $\mathcal{L}$ and elements of $L(\mathcal{L})$. In particular, $\langle F, \models \rangle$ is ordinary first-order logic, i.e., F is the function assigning to the language $\mathcal{L}$ the set of first-order sentences built from the constants in $\mathcal{L}$, and $\models$ is the satisfaction relation of first-order logic. mod:lin:alg:sec

Notice that we are still employing the same notion of structure for a given language as for first-order logic, but we do not presuppose that sentences are build up from the basic symbols in $\mathcal{L}$ in the usual way, nor that the relation $\models_L$ is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where sentences in $\langle L, \models_L \rangle$ contain infinitely long conjunctions or disjunction, or quantifiers other than $\exists$ and $\forall$ (e.g., "there are infinitely many x such that ..."), or perhaps infinitely long quantifier prefixes. To emphasize that "sentences" in $L(\mathcal{L})$ need not be ordinary sentences of first-order logic, in this chapter we use variables α, β, ... to range over them, and reserve φ, ψ, ... for ordinary first-order formulas.

Definition 4.2. Let $\mathrm{Mod}_L(\alpha)$ denote the class $\{\mathfrak{M} : \mathfrak{M} \models_L \alpha\}$. If the language needs to be made explicit, we write $\mathrm{Mod}_L^{\mathcal{L}}(\alpha)$. Two structures $\mathfrak{M}$ and $\mathfrak{N}$ for $\mathcal{L}$ are *elementarily equivalent in* $\langle L, \models_L\rangle$, written $\mathfrak{M} \equiv_L \mathfrak{N}$, if the same sentences from $L(\mathcal{L})$ are true in each.

Definition 4.3. An abstract logic $\langle L, \models_L\rangle$ for the language $\mathcal{L}$ is *normal* if it satisfies the following properties:

1. (*L-Monotonicity*) For languages $\mathcal{L}$ and $\mathcal{L}'$, if $\mathcal{L} \subseteq \mathcal{L}'$, then $L(\mathcal{L}) \subseteq L(\mathcal{L}')$.

2. (*Expansion Property*) For each $\alpha \in L(\mathcal{L})$ there is a *finite* subset $\mathcal{L}'$ of $\mathcal{L}$ such that the relation $\mathfrak{M} \models_L \alpha$ depends only on the reduct of $\mathfrak{M}$ to $\mathcal{L}'$; i.e., if $\mathfrak{M}$ and $\mathfrak{N}$ have the same reduct to $\mathcal{L}'$ then $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{N} \models_L \alpha$.

3. (*Isomorphism Property*) If $\mathfrak{M} \models_L \alpha$ and $\mathfrak{M} \simeq \mathfrak{N}$ then also $\mathfrak{N} \models_L \alpha$.

4. (*Renaming Property*) The relation $\models_L$ is preserved under renaming: if the language $\mathcal{L}'$ is obtained from $\mathcal{L}$ by replacing each symbol P by a symbol P' of the same arity and each constant c by a distinct constant c', then for each structure $\mathfrak{M}$ and sentence α, $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{M}' \models_L \alpha'$, where $\mathfrak{M}'$ is the $\mathcal{L}'$-structure corresponding to $\mathcal{L}$ and $\alpha' \in L(\mathcal{L}')$.

5. (*Boolean Property*) The abstract logic $\langle L, \models_L\rangle$ is closed under the Boolean connectives in the sense that for each $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that $\mathfrak{M} \models_L \beta$ if and only if $\mathfrak{M} \not\models_L \alpha$, and for each α and β there is a γ such that $\mathrm{Mod}_L(\gamma) = \mathrm{Mod}_L(\alpha) \cap \mathrm{Mod}_L(\beta)$. Similarly for atomic formulas and the other connectives.

6. (*Quantifier Property*) For each constant c in $\mathcal{L}$ and $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that

$$\mathrm{Mod}_L^{\mathcal{L}'}(\beta) = \{\mathfrak{M} : (\mathfrak{M}, a)\} \in \mathrm{Mod}_L^{\mathcal{L}}(\alpha) \text{ for some } a \in |\mathfrak{M}|\},$$

   where $\mathcal{L}' = \mathcal{L} \setminus \{c\}$ and $(\mathfrak{M}, a)$ is the expansion of $\mathfrak{M}$ to $\mathcal{L}$ assigning a to c.

7. (*Relativization Property*) Given a sentence $\alpha \in L(\mathcal{L})$ and symbols R, c_1, $\dots$, c_n not in $\mathcal{L}$, there is a sentence $\beta \in L(\mathcal{L} \cup \{R, c_1, \dots, c_n\})$ called the *relativization* of α to $R(x, c_1, \dots c_n)$, such that for each structure $\mathfrak{M}$:

$$(\mathfrak{M}, X, b_1, \dots, b_n) \models_L \beta) \text{ if and only if } \mathfrak{N} \models_L \alpha,$$

   where $\mathfrak{N}$ is the substructure of $\mathfrak{M}$ with domain $|\mathfrak{N}| = \{a \in |\mathfrak{M}| : R^{\mathfrak{M}}(a, b_1, \dots, b_n)\}$ (see Remark 1), and $(\mathfrak{M}, X, b_1, \dots, b_n)$ is the expansion of $\mathfrak{M}$ interpreting R, c_1, $\dots$, c_n by X, b_1, $\dots$, b_n, respectively (with $X \subseteq M^{n+1}$).

Definition 4.4. Given two abstract logics $\langle L_1, \models_{L_1}\rangle$ and $\langle L_2, \models_{L_2}\rangle$ we say that the latter is *at least as expressive* as the former, written $\langle L_1, \models_{L_1}\rangle \leq \langle L_2, \models_{L_2}\rangle$, if for each language $\mathcal{L}$ and sentence $\alpha \in L_1(\mathcal{L})$ there is a sentence $\beta \in L_2(\mathcal{L})$ such that $\mathrm{Mod}^{\mathcal{L}}_{L_1}(\alpha) = \mathrm{Mod}^{\mathcal{L}}_{L_2}(\beta)$. The logics $\langle L_1, \models_{L_1}\rangle$ and $\langle L_2, \models_{L_2}\rangle$ are *equivalent* if $\langle L_1, \models_{L_1}\rangle \leq \langle L_2, \models_{L_2}\rangle$ and $\langle L_2, \models_{L_2}\rangle \leq \langle L_1, \models_{L_1}\rangle$.

Remark 5. First-order logic, i.e., the abstract logic $\langle F, \models\rangle$, is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a sentence α to $R(x, c_1, \ldots, c_n)$ is obtained by replacing each subformula $\forall x\, \beta$ by $\forall x\, (R(x, c_1, \ldots, c_n) \to \beta)$. Moreover, if $\langle L, \models_L\rangle$ is normal, then $\langle F, \models\rangle \leq \langle L, \models_L\rangle$, as can be can shown by induction on first-order formulas. Accordingly, with no loss in generality, we can assume that every first-order sentence belongs to every normal logic.

4.3 Compactness and Löwenheim–Skolem Properties

We now give the obvious extensions of compactness and Löwenheim–Skolem to the case of abstract logics. mod:lin:lsp:sec

Definition 4.5. An abstract logic $\langle L, \models_L\rangle$ has the *Compactness Property* if each set Γ of $L(\mathcal{L})$-sentences is satisfiable whenever each finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

Definition 4.6. $\langle L, \models_L\rangle$ has the *Downward Löwenheim–Skolem property* if any satisfiable Γ has an enumerable model.

The notion of partial isomorphism from Definition 1.15 is purely "algebraic" (i.e., given without reference to the sentences of the language but only to the constants provided by the language $\mathcal{L}$ of the structures), and hence it applies to the case of abstract logics. In case of first-order logic, we know from Theorem 1.17 that if two structures are partially isomorphic then they are elementarily equivalent. That proof does not carry over to abstract logics, for induction on formulas need not be available for arbitrary $\alpha \in L(\mathcal{L})$, but the theorem is true nonetheless, provided the Löwenheim–Skolem property holds.

Theorem 4.7. *Suppose $\langle L, \models_L\rangle$ is a normal logic with the Löwenheim–Skolem property. Then any two structures that are partially isomorphic are elementarily equivalent in $\langle L, \models_L\rangle$.* mod:lin:lsp:thm:abstract-p-isom

Proof. Suppose $\mathfrak{M} \simeq_p \mathfrak{N}$, but for some α also $\mathfrak{M} \models_L \alpha$ while $\mathfrak{N} \not\models_L \alpha$. By the Isomorphism Property we can assume that $|\mathfrak{M}|$ and $|\mathfrak{N}|$ are disjoint, and by the Expansion Property we can assume that $\alpha \in L(\mathcal{L})$ for a finite language $\mathcal{L}$. Let $\mathcal{I}$ be a set of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$, and with no loss of generality also assume that if $p \in \mathcal{I}$ and $q \subseteq p$ then also $q \in \mathcal{I}$.

$|\mathfrak{M}|^{<\omega}$ is the set of finite sequences of elements of $|\mathfrak{M}|$. Let S be the ternary relation over $|\mathfrak{M}|^{<\omega}$ representing concatenation, i.e., if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ then

$S(\mathbf{a}, \mathbf{b}, \mathbf{c})$ holds if and only if $\mathbf{c}$ is the concatenation of $\mathbf{a}$ and $\mathbf{b}$; and let T be the ternary relation such that $T(\mathbf{a}, b, \mathbf{c})$ holds for $b \in M$ and $\mathbf{a}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ if and only if $\mathbf{a} = a_1, \dots a_n$ and $\mathbf{c} = a_1, \dots a_n, b$. Pick new 3-place predicate symbols P and Q and form the structure $\mathfrak{M}^*$ having the universe $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, having $\mathfrak{M}$ as a substructure, and interpreting P and Q by the concatenation relations S and T (so $\mathfrak{M}^*$ is in the language $\mathcal{L} \cup \{P, Q\}$).

Define $|\mathfrak{N}|^{<\omega}$, S', T', P', Q' and $\mathfrak{N}^*$ analogously. Since by hypothesis $\mathfrak{M} \simeq_p \mathfrak{N}$, there is a relation I between $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ such that $I(\mathbf{a}, \mathbf{b})$ holds if and only if $\mathbf{a}$ and $\mathbf{b}$ are isomorphic and satisfy the back-and-forth condition of Definition 1.15. Now, let $\mathfrak{M}$ be the structure whose domain is the union of the domains of $\mathfrak{M}^*$ and $\mathfrak{N}^*$, having $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as substructures, in the language with one extra binary predicate symbol R interpreted by the relation I and predicate symbols denoting the domains $|\mathfrak{M}|^*$ and $|\mathfrak{N}|*$.

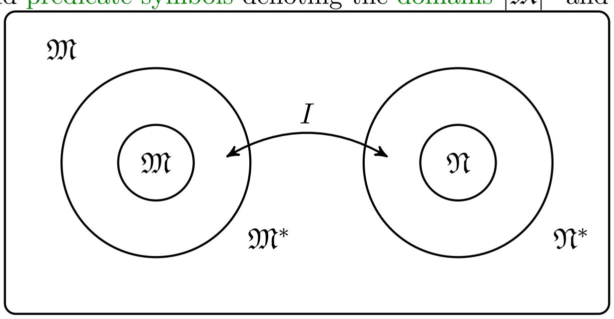

Figure 4.1: The structure $\mathfrak{M}$ with the internal partial isomorphism.

The crucial observation is that in the language of the structure $\mathfrak{M}$ there is a *first-order* sentence θ_1 true in $\mathfrak{M}$ saying that $\mathfrak{M} \models_L \alpha$ and $\mathfrak{N} \not\models_L \alpha$ (this requires the Relativization Property), as well as a *first-order* sentence θ_2 true in $\mathfrak{M}$ saying that $\mathfrak{M} \simeq_p \mathfrak{N}$ via the partial isomorphism I. By the Löwenheim–Skolem Property, θ_1 and θ_2 are jointly true in an enumerable model $\mathfrak{M}_0$ containing partially isomorphic substructures $\mathfrak{M}_0$ and $\mathfrak{N}_0$ such that $\mathfrak{M}_0 \models_L \alpha$ and $\mathfrak{N}_0 \not\models_L \alpha$. But enumerable partially isomorphic structures are in fact isomorphic by Theorem 1.16, contradicting the Isomorphism Property of normal abstract logics. □

4.4 Lindström's Theorem

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Lemma 4.8. *Suppose $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, and assume also that there is an $n \in \mathbb{N}$ such that for any two structures $\mathfrak{M}$ and $\mathfrak{N}$, if $\mathfrak{M} \equiv_n \mathfrak{N}$ and $\mathfrak{M} \models_L \alpha$ then also $\mathfrak{N} \models_L \alpha$. Then α is equivalent to a first-order sentence, i.e., there is a first-order θ such that $\mathrm{Mod}_L(\alpha) = \mathrm{Mod}_L(\theta)$.*

Proof. Let n be such that any two n-equivalent structures $\mathfrak{M}$ and $\mathfrak{N}$ agree on the value assigned to α. Recall Proposition 1.19: there are only finitely many first-order sentences in a finite language that have quantifier rank no greater than n, up to logical equivalence. Now, for each fixed structure $\mathfrak{M}$ let $\theta_{\mathfrak{M}}$ be the conjunction of all first-order sentences α true in $\mathfrak{M}$ with $\mathrm{qr}(\alpha) \leq n$ (this conjunction is finite), so that $\mathfrak{N} \models \theta_{\mathfrak{M}}$ if and only if $\mathfrak{N} \equiv_n \mathfrak{M}$. Then put $\theta = \bigvee\{\theta_{\mathfrak{M}} : \mathfrak{M} \models_L \alpha\}$; this disjunction is also finite (up to logical equivalence).

The conclusion $\mathrm{Mod}_L(\alpha) = \mathrm{Mod}_L(\theta)$ follows. In fact, if $\mathfrak{N} \models_L \theta$ then for some $\mathfrak{M} \models_L \alpha$ we have $\mathfrak{N} \models \theta_{\mathfrak{M}}$, whence also $\mathfrak{N} \models_L \alpha$ (by the hypothesis of the lemma). Conversely, if $\mathfrak{N} \models_L \alpha$ then $\theta_{\mathfrak{N}}$ is a disjunct in θ, and since $\mathfrak{N} \models \theta_{\mathfrak{N}}$, also $\mathfrak{N} \models_L \theta$. □

Theorem 4.9 (Lindström's Theorem). *Suppose $\langle L, \models_L\rangle$ has the Compactness and the Löwenheim–Skolem Properties. Then $\langle L, \models_L\rangle \leq \langle F, \models\rangle$ (so $\langle L, \models_L \rangle$ is equivalent to first-order logic).* mod:lin:prf:thm:lindstrom

Proof. By Lemma 4.8, it suffices to show that for any $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, there is $n \in \mathbb{N}$ such that for any two structures $\mathfrak{M}$ and $\mathfrak{N}$: if $\mathfrak{M} \equiv_n \mathfrak{N}$ then $\mathfrak{M}$ and $\mathfrak{N}$ agree on α. For then α is equivalent to a first-order sentence, from which $\langle L, \models_L\rangle \leq \langle F, \models\rangle$ follows. Since we are working in a finite, purely relational language, by Theorem 1.23 we can replace the statement that $\mathfrak{M} \equiv_n \mathfrak{N}$ by the corresponding algebraic statement that $I_n(\emptyset, \emptyset)$.

Given α, suppose towards a contradiction that for each n there are structures $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that $I_n(\emptyset, \emptyset)$, but (say) $\mathfrak{M}_n \models_L \alpha$ whereas $\mathfrak{N}_n \not\models_L \alpha$. By the Isomorphism Property we can assume that all the $\mathfrak{M}_n$'s interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic sentences in the language, we may also assume that they satisfy the same atomic sentences (we can take a subsequence of the $\mathfrak{M}$'s otherwise). Let $\mathfrak{M}$ be the union of all the $\mathfrak{M}_n$'s, i.e., the unique minimal structure having each $\mathfrak{M}_n$ as a substructure. As in the proof of Theorem 4.7, let $\mathfrak{M}^*$ be the extension of $\mathfrak{M}$ with domain $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, in the expanded language comprising the concatenation predicates P and Q.

Similarly, define $\mathfrak{N}_n$, $\mathfrak{N}$ and $\mathfrak{N}^*$. Now let $\mathfrak{M}$ be the structure whose domain comprises the domains of $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as well as the natural numbers $\mathbb{N}$ along with their natural ordering $\leq$, in the language with extra predicates representing the domains $|\mathfrak{M}|$, $|\mathfrak{N}|$, $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ as well as predicates coding the domains of $\mathfrak{M}_n$ and $\mathfrak{N}_n$ in the sense that:

$$|\mathfrak{M}_n| = \{a \in |\mathfrak{M}| : R(a, n)\}; \qquad |\mathfrak{N}_n| = \{a \in |\mathfrak{N}| : S(a, n)\};$$
$$|\mathfrak{M}|^{<\omega}_n = \{a \in |\mathfrak{M}|^{<\omega} : R(a, n)\}; \qquad |\mathfrak{N}|^{<\omega}_n = \{a \in |\mathfrak{N}|^{<\omega} : S(a, n)\}.$$

The structure $\mathfrak{M}$ also has a ternary relation J such that $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Now there is a sentence θ in the language $\mathcal{L}$ augmented by R, S, J, etc., saying that $\leq$ is a discrete linear ordering with first but no last element and such that $\mathfrak{M}_n \models \alpha$, $\mathfrak{N}_n \not\models \alpha$, and for each n in the ordering, $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Using the Compactness Property, we can find a model $\mathfrak{M}^*$ of θ in which the ordering contains a non-standard element n^*. In particular then $\mathfrak{M}^*$ will contain substructures $\mathfrak{M}_{n^*}$ and $\mathfrak{N}_{n^*}$ such that $\mathfrak{M}_{n^*} \models_L \alpha$ and $\mathfrak{N}_{n^*} \not\models_L \alpha$. But now we can define a set $\mathcal{I}$ of pairs of k-tuples from $|\mathfrak{M}_{n^*}|$ and $|\mathfrak{N}_{n^*}|$ by putting $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathcal{I}$ if and only if $J(n^* - k, \mathbf{a}, \mathbf{b})$, where k is the length of $\mathbf{a}$ and $\mathbf{b}$. Since n^* is non-standard, for each standard k we have that $n^* - k > 0$, and the set $\mathcal{I}$ witnesses the fact that $\mathfrak{M}_{n^*} \simeq_p \mathfrak{N}_{n^*}$. But by Theorem 4.7, $\mathfrak{M}_{n^*}$ is L-equivalent to $\mathfrak{N}_{n^*}$, a contradiction. □

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Bibliography