

lin.1 Compactness and Löwenheim-Skolem Properties

mod:lin:lsp:
sec We now give the obvious extensions of compactness and Löwenheim-Skolem to the case of abstract logics.

Definition lin.1. An abstract logic $\langle L, \models_L \rangle$ has the *Compactness Property* if each set Γ of $L(\mathcal{L})$ -sentences is satisfiable whenever each finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

Definition lin.2. $\langle L, \models_L \rangle$ has the *Downward Löwenheim-Skolem property* if any satisfiable Γ has an **enumerable** model.

The notion of partial isomorphism from ?? is purely “algebraic” (i.e., given without reference to the **sentences** of the language but only to the constants provided by the **language** \mathcal{L} of the **structures**), and hence it applies to the case of abstract logics. In case of first-order logic, we know from ?? that if two **structures** are partially isomorphic then they are elementarily equivalent. That proof does not carry over to abstract logics, for induction on **formulas** need not be available for arbitrary $\alpha \in L(\mathcal{L})$, but the theorem is true nonetheless, provided the Löwenheim-Skolem property holds.

mod:lin:lsp:
thm:abstract-p-isom **Theorem lin.3.** *Suppose $\langle L, \models_L \rangle$ is a normal logic with the Löwenheim-Skolem property. Then any two **structures** that are partially isomorphic are elementarily equivalent in $\langle L, \models_L \rangle$.*

Proof. Suppose $\mathfrak{M} \simeq_p \mathfrak{N}$, but for some α also $\mathfrak{M} \models_L \alpha$ while $\mathfrak{N} \not\models_L \alpha$. By the Isomorphism Property we can assume that $|\mathfrak{M}|$ and $|\mathfrak{N}|$ are disjoint, and by the Expansion Property we can assume that $\alpha \in L(\mathcal{L})$ for a finite **language** \mathcal{L} . Let \mathcal{I} be a set of partial isomorphisms between \mathfrak{M} and \mathfrak{N} , and with no loss of generality also assume that if $p \in \mathcal{I}$ and $q \subseteq p$ then also $q \in \mathcal{I}$.

$|\mathfrak{M}|^{<\omega}$ is the set of finite sequences of **elements** of $|\mathfrak{M}|$. Let S be the ternary relation over $|\mathfrak{M}|^{<\omega}$ representing concatenation, i.e., if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ then $S(\mathbf{a}, \mathbf{b}, \mathbf{c})$ holds if and only if \mathbf{c} is the concatenation of \mathbf{a} and \mathbf{b} ; and let T be the ternary relation such that $T(\mathbf{a}, b, \mathbf{c})$ holds for $b \in M$ and $\mathbf{a}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ if and only if $\mathbf{a} = a_1, \dots, a_n$ and $\mathbf{c} = a_1, \dots, a_n, b$. Pick new 3-place **predicate symbols** P and Q and form the **structure** \mathfrak{M}^* having the universe $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, having \mathfrak{M} as a substructure, and interpreting P and Q by the concatenation relations S and T (so \mathfrak{M}^* is in the **language** $\mathcal{L} \cup \{P, Q\}$).

Define $|\mathfrak{N}|^{<\omega}$, S' , T' , P' , Q' and \mathfrak{N}^* analogously. Since by hypothesis $\mathfrak{M} \simeq_p \mathfrak{N}$, there is a relation I between $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ such that $I(\mathbf{a}, \mathbf{b})$ holds if and only if \mathbf{a} and \mathbf{b} are isomorphic and satisfy the back-and-forth condition of ??. Now, let \mathfrak{M} be the **structure** whose **domain** is the union of the **domains** of \mathfrak{M}^* and \mathfrak{N}^* , having \mathfrak{M}^* and \mathfrak{N}^* as **substructures**, in the **language** with one extra binary **predicate symbol** R interpreted by the relation I and **predicate symbols** denoting the **domains** $|\mathfrak{M}|^*$ and $|\mathfrak{N}|^*$.

The crucial observation is that in the **language** of the **structure** \mathfrak{M} there is a *first-order sentence* θ_1 true in \mathfrak{M} saying that $\mathfrak{M} \models_L \alpha$ and $\mathfrak{N} \not\models_L \alpha$ (this requires the Relativization Property), as well as a *first-order sentence*

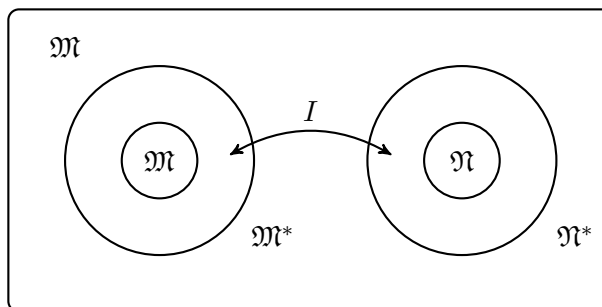


Figure 1: The structure \mathfrak{M} with the internal partial isomorphism.

θ_2 true in \mathfrak{M} saying that $\mathfrak{M} \simeq_p \mathfrak{N}$ via the partial isomorphism I . By the Löwenheim-Skolem Property, θ_1 and θ_2 are jointly true in an enumerable model \mathfrak{M}_0 containing partially isomorphic substructures \mathfrak{M}_0 and \mathfrak{N}_0 such that $\mathfrak{M}_0 \models_L \alpha$ and $\mathfrak{N}_0 \not\models_L \alpha$. But enumerable partially isomorphic structures are in fact isomorphic by ??, contradicting the Isomorphism Property of normal abstract logics. \square

Photo Credits

Bibliography