

Chapter udf

Lindström's Theorem

lin.1 Introduction

In this chapter we aim to prove Lindström's characterization of first-order logic as the maximal logic for which (given certain further constraints) the Compactness and the Downward Löwenheim-Skolem theorems hold (?? and ??). First, we need a more general characterization of the general class of logics to which the theorem applies. We will restrict ourselves to *relational* languages, i.e., languages which only contain **predicate symbols** and individual constants, but no **function symbols**.

lin.2 Abstract Logics

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Definition lin.1. An *abstract logic* is a pair $\langle L, \models_L \rangle$, where L is a function that assigns to each **language** \mathcal{L} a set $L(\mathcal{L})$ of **sentences**, and \models_L is a relation between **structures** for the **language** \mathcal{L} and **elements** of $L(\mathcal{L})$. In particular, $\langle F, \models \rangle$ is ordinary first-order logic, i.e., F is the function assigning to the **language** \mathcal{L} the set of first-order **sentences** built from the constants in \mathcal{L} , and \models is the satisfaction relation of first-order logic.

Notice that we are still employing the same notion of **structure** for a given **language** as for first-order logic, but we do not presuppose that **sentences** are built up from the basic symbols in \mathcal{L} in the usual way, nor that the relation \models_L is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where **sentences** in $\langle L, \models_L \rangle$ contain infinitely long conjunctions or disjunction, or quantifiers other than \exists and \forall (e.g., “there are infinitely many x such that ...”), or perhaps infinitely long quantifier prefixes. To emphasize that “**sentences**” in $L(\mathcal{L})$ need not be ordinary **sentences** of first-order logic, in this chapter we use **variables** α, β, \dots to range over them, and reserve φ, ψ, \dots for ordinary first-order **formulas**.

Definition lin.2. Let $\text{Mod}_L(\alpha)$ denote the class $\{\mathfrak{M} : \mathfrak{M} \models_L \alpha\}$. If the **language** needs to be made explicit, we write $\text{Mod}_L^{\mathcal{L}}(\alpha)$. Two **structures** \mathfrak{M} and \mathfrak{N} for \mathcal{L} are *elementarily equivalent in* $\langle L, \models_L \rangle$, written $\mathfrak{M} \equiv_L \mathfrak{N}$, if the same **sentences** from $L(\mathcal{L})$ are true in each.

Definition lin.3. An abstract logic $\langle L, \models_L \rangle$ for the **language** \mathcal{L} is *normal* if it satisfies the following properties:

1. (*L-Monotonicity*) For **languages** \mathcal{L} and \mathcal{L}' , if $\mathcal{L} \subseteq \mathcal{L}'$, then $L(\mathcal{L}) \subseteq L(\mathcal{L}')$.
2. (*Expansion Property*) For each $\alpha \in L(\mathcal{L})$ there is a *finite* subset \mathcal{L}' of \mathcal{L} such that the relation $\mathfrak{M} \models_L \alpha$ depends only on the reduct of \mathfrak{M} to \mathcal{L}' ; i.e., if \mathfrak{M} and \mathfrak{N} have the same reduct to \mathcal{L}' then $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{N} \models_L \alpha$.
3. (*Isomorphism Property*) If $\mathfrak{M} \models_L \alpha$ and $\mathfrak{M} \simeq \mathfrak{N}$ then also $\mathfrak{N} \models_L \alpha$.
4. (*Renaming Property*) The relation \models_L is preserved under renaming: if the **language** \mathcal{L}' is obtained from \mathcal{L} by replacing each symbol P by a symbol P' of the same arity and each constant c by a distinct constant c' , then for each **structure** \mathfrak{M} and **sentence** α , $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{M}' \models_L \alpha'$, where \mathfrak{M}' is the \mathcal{L}' -**structure** corresponding to \mathfrak{M} and $\alpha' \in L(\mathcal{L}')$.
5. (*Boolean Property*) The abstract logic $\langle L, \models_L \rangle$ is closed under the Boolean connectives in the sense that for each $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that $\mathfrak{M} \models_L \beta$ if and only if $\mathfrak{M} \not\models_L \alpha$, and for each α and β there is a γ such that $\text{Mod}_L(\gamma) = \text{Mod}_L(\alpha) \cap \text{Mod}_L(\beta)$. Similarly for atomic **formulas** and the other connectives.
6. (*Quantifier Property*) For each constant c in \mathcal{L} and $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that

$$\text{Mod}_L^{\mathcal{L}'}(\beta) = \{\mathfrak{M} : (\mathfrak{M}, a) \in \text{Mod}_L^{\mathcal{L}}(\alpha) \text{ for some } a \in |\mathfrak{M}|\},$$

where $\mathcal{L}' = \mathcal{L} \setminus \{c\}$ and (\mathfrak{M}, a) is the expansion of \mathfrak{M} to \mathcal{L} assigning a to c .

7. (*Relativization Property*) Given a **sentence** $\alpha \in L(\mathcal{L})$ and symbols R, c_1, \dots, c_n not in \mathcal{L} , there is a **sentence** $\beta \in L(\mathcal{L} \cup \{R, c_1, \dots, c_n\})$ called the *relativization* of α to $R(x, c_1, \dots, c_n)$, such that for each **structure** \mathfrak{M} :

$$(\mathfrak{M}, X, b_1, \dots, b_n) \models_L \beta \text{ if and only if } \mathfrak{N} \models_L \alpha,$$

where \mathfrak{N} is the substructure of \mathfrak{M} with **domain** $|\mathfrak{N}| = \{a \in |\mathfrak{M}| : R^{\mathfrak{M}}(a, b_1, \dots, b_n)\}$ (see ??), and $(\mathfrak{M}, X, b_1, \dots, b_n)$ is the expansion of \mathfrak{M} interpreting R, c_1, \dots, c_n by X, b_1, \dots, b_n , respectively (with $X \subseteq M^{n+1}$).

Definition lin.4. Given two abstract logics $\langle L_1, \models_{L_1} \rangle$ and $\langle L_2, \models_{L_2} \rangle$ we say that the latter is *at least as expressive* as the former, written $\langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle$, if for each **language** \mathcal{L} and **sentence** $\alpha \in L_1(\mathcal{L})$ there is a **sentence** $\beta \in L_2(\mathcal{L})$ such that $\text{Mod}_{L_1}^{\mathcal{L}}(\alpha) = \text{Mod}_{L_2}^{\mathcal{L}}(\beta)$. The logics $\langle L_1, \models_{L_1} \rangle$ and $\langle L_2, \models_{L_2} \rangle$ are *equivalent* if $\langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle$ and $\langle L_2, \models_{L_2} \rangle \leq \langle L_1, \models_{L_1} \rangle$.

Remark 1. First-order logic, i.e., the abstract logic $\langle F, \models \rangle$, is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a **sentence** α to $R(x, c_1, \dots, c_n)$ is obtained by replacing each **subformula** $\forall x \beta$ by $\forall x (R(x, c_1, \dots, c_n) \rightarrow \beta)$. Moreover, if $\langle L, \models_L \rangle$ is normal, then $\langle F, \models \rangle \leq \langle L, \models_L \rangle$, as can be shown by induction on first-order **formulas**. Accordingly, with no loss in generality, we can assume that every first-order **sentence** belongs to every normal logic.

lin.3 Compactness and Löwenheim-Skolem Properties

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sec We now give the obvious extensions of compactness and Löwenheim-Skolem to the case of abstract logics.

Definition lin.5. An abstract logic $\langle L, \models_L \rangle$ has the *Compactness Property* if each set Γ of $L(\mathcal{L})$ -**sentences** is satisfiable whenever each finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

Definition lin.6. $\langle L, \models_L \rangle$ has the *Downward Löwenheim-Skolem property* if any satisfiable Γ has an **enumerable** model.

The notion of partial isomorphism from ?? is purely “algebraic” (i.e., given without reference to the **sentences** of the language but only to the constants provided by the **language** \mathcal{L} of the **structures**), and hence it applies to the case of abstract logics. In case of first-order logic, we know from ?? that if two **structures** are partially isomorphic then they are elementarily equivalent. That proof does not carry over to abstract logics, for induction on **formulas** need not be available for arbitrary $\alpha \in L(\mathcal{L})$, but the theorem is true nonetheless, provided the Löwenheim-Skolem property holds.

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thm:abstract-p-isom **Theorem lin.7.** *Suppose $\langle L, \models_L \rangle$ is a normal logic with the Löwenheim-Skolem property. Then any two **structures** that are partially isomorphic are elementarily equivalent in $\langle L, \models_L \rangle$.*

Proof. Suppose $\mathfrak{M} \simeq_p \mathfrak{N}$, but for some α also $\mathfrak{M} \models_L \alpha$ while $\mathfrak{N} \not\models_L \alpha$. By the Isomorphism Property we can assume that $|\mathfrak{M}|$ and $|\mathfrak{N}|$ are disjoint, and by the Expansion Property we can assume that $\alpha \in L(\mathcal{L})$ for a finite **language** \mathcal{L} . Let \mathcal{I} be a set of partial isomorphisms between \mathfrak{M} and \mathfrak{N} , and with no loss of generality also assume that if $p \in \mathcal{I}$ and $q \subseteq p$ then also $q \in \mathcal{I}$.

$|\mathfrak{M}|^{<\omega}$ is the set of finite sequences of **elements** of $|\mathfrak{M}|$. Let S be the ternary relation over $|\mathfrak{M}|^{<\omega}$ representing concatenation, i.e., if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ then

$S(\mathbf{a}, \mathbf{b}, \mathbf{c})$ holds if and only if \mathbf{c} is the concatenation of \mathbf{a} and \mathbf{b} ; and let T be the ternary relation such that $T(\mathbf{a}, b, \mathbf{c})$ holds for $b \in M$ and $\mathbf{a}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ if and only if $\mathbf{a} = a_1, \dots, a_n$ and $\mathbf{c} = a_1, \dots, a_n, b$. Pick new 3-place predicate symbols P and Q and form the structure \mathfrak{M}^* having the universe $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, having \mathfrak{M} as a substructure, and interpreting P and Q by the concatenation relations S and T (so \mathfrak{M}^* is in the language $\mathcal{L} \cup \{P, Q\}$).

Define $|\mathfrak{N}|^{<\omega}$, S' , T' , P' , Q' and \mathfrak{N}^* analogously. Since by hypothesis $\mathfrak{M} \simeq_p \mathfrak{N}$, there is a relation I between $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ such that $I(\mathbf{a}, \mathbf{b})$ holds if and only if \mathbf{a} and \mathbf{b} are isomorphic and satisfy the back-and-forth condition of ???. Now, let \mathfrak{M} be the structure whose domain is the union of the domains of \mathfrak{M}^* and \mathfrak{N}^* , having \mathfrak{M}^* and \mathfrak{N}^* as substructures, in the language with one extra binary predicate symbol R interpreted by the relation I and predicate symbols denoting the domains $|\mathfrak{M}^*|$ and $|\mathfrak{N}^*|$.

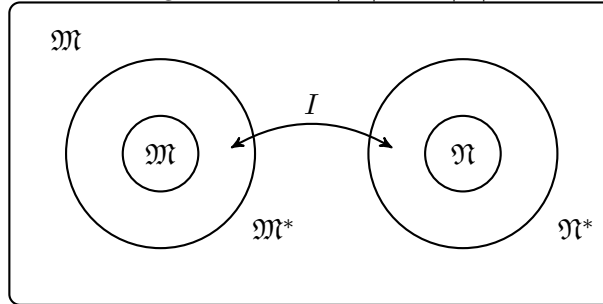


Figure lin.1: The structure \mathfrak{M} with the internal partial isomorphism.

The crucial observation is that in the language of the structure \mathfrak{M} there is a first-order sentence θ_1 true in \mathfrak{M} saying that $\mathfrak{M} \models_L \alpha$ and $\mathfrak{N} \not\models_L \alpha$ (this requires the Relativization Property), as well as a first-order sentence θ_2 true in \mathfrak{M} saying that $\mathfrak{M} \simeq_p \mathfrak{N}$ via the partial isomorphism I . By the Löwenheim-Skolem Property, θ_1 and θ_2 are jointly true in an enumerable model \mathfrak{M}_0 containing partially isomorphic substructures \mathfrak{M}_0 and \mathfrak{N}_0 such that $\mathfrak{M}_0 \models_L \alpha$ and $\mathfrak{N}_0 \not\models_L \alpha$. But enumerable partially isomorphic structures are in fact isomorphic by ??, contradicting the Isomorphism Property of normal abstract logics. \square

lin.4 Lindström's Theorem

Lemma lin.8. *Suppose $\alpha \in L(\mathcal{L})$, with \mathcal{L} finite, and assume also that there is an $n \in \mathbb{N}$ such that for any two structures \mathfrak{M} and \mathfrak{N} , if $\mathfrak{M} \equiv_n \mathfrak{N}$ and $\mathfrak{M} \models_L \alpha$ then also $\mathfrak{N} \models_L \alpha$. Then α is equivalent to a first-order sentence, i.e., there is a first-order θ such that $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$.*

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Proof. Let n be such that any two n -equivalent structures \mathfrak{M} and \mathfrak{N} agree on the value assigned to α . Recall ???: there are only finitely many first-order sentences in a finite language that have quantifier rank no greater than n , up to

logical equivalence. Now, for each fixed **structure** \mathfrak{M} let $\theta_{\mathfrak{M}}$ be the conjunction of all first-order **sentences** α true in \mathfrak{M} with $\text{qr}(\alpha) \leq n$ (this conjunction is finite), so that $\mathfrak{N} \models \theta_{\mathfrak{M}}$ if and only if $\mathfrak{N} \equiv_n \mathfrak{M}$. Then put $\theta = \bigvee \{\theta_{\mathfrak{M}} : \mathfrak{M} \models_L \alpha\}$; this disjunction is also finite (up to logical equivalence).

The conclusion $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$ follows. In fact, if $\mathfrak{N} \models_L \theta$ then for some $\mathfrak{M} \models_L \alpha$ we have $\mathfrak{N} \models \theta_{\mathfrak{M}}$, whence also $\mathfrak{N} \models_L \alpha$ (by the hypothesis of the lemma). Conversely, if $\mathfrak{N} \models_L \alpha$ then $\theta_{\mathfrak{N}}$ is a disjunct in θ , and since $\mathfrak{N} \models \theta_{\mathfrak{N}}$, also $\mathfrak{N} \models_L \theta$. \square

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Theorem lin.9 (Lindström's Theorem). *Suppose $\langle L, \models_L \rangle$ has the Compactness and the Löwenheim-Skolem Properties. Then $\langle L, \models_L \rangle \leq \langle F, \models \rangle$ (so $\langle L, \models_L \rangle$ is equivalent to first-order logic).*

Proof. By **Lemma lin.8**, it suffices to show that for any $\alpha \in L(\mathcal{L})$, with \mathcal{L} finite, there is $n \in \mathbb{N}$ such that for any two **structures** \mathfrak{M} and \mathfrak{N} : if $\mathfrak{M} \equiv_n \mathfrak{N}$ then \mathfrak{M} and \mathfrak{N} agree on α . For then α is equivalent to a first-order **sentence**, from which $\langle L, \models_L \rangle \leq \langle F, \models \rangle$ follows. Since we are working in a finite, purely relational **language**, by ?? we can replace the statement that $\mathfrak{M} \equiv_n \mathfrak{N}$ by the corresponding algebraic statement that $I_n(\emptyset, \emptyset)$.

Given α , suppose towards a contradiction that for each n there are **structures** \mathfrak{M}_n and \mathfrak{N}_n such that $I_n(\emptyset, \emptyset)$, but (say) $\mathfrak{M}_n \models_L \alpha$ whereas $\mathfrak{N}_n \not\models_L \alpha$. By the Isomorphism Property we can assume that all the \mathfrak{M}_n 's interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic **sentences** in the language, we may also assume that they satisfy the same atomic **sentences** (we can take a subsequence of the \mathfrak{M} 's otherwise). Let \mathfrak{M} be the union of all the \mathfrak{M}_n 's, i.e., the unique minimal **structure** having each \mathfrak{M}_n as a substructure. As in the proof of **Theorem lin.7**, let \mathfrak{M}^* be the extension of \mathfrak{M} with **domain** $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, in the expanded **language** comprising the concatenation predicates P and Q .

Similarly, define \mathfrak{N}_n , \mathfrak{N} and \mathfrak{N}^* . Now let \mathfrak{M} be the **structure** whose **domain** comprises the **domains** of \mathfrak{M}^* and \mathfrak{N}^* as well as the natural numbers \mathbb{N} along with their natural ordering \leq , in the **language** with extra predicates representing the **domains** $|\mathfrak{M}|$, $|\mathfrak{N}|$, $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ as well as predicates coding the domains of \mathfrak{M}_n and \mathfrak{N}_n in the sense that:

$$\begin{aligned} |\mathfrak{M}_n| &= \{a \in |\mathfrak{M}| : R(a, n)\}; & |\mathfrak{N}_n| &= \{a \in |\mathfrak{N}| : S(a, n)\}; \\ |\mathfrak{M}_n|^{<\omega} &= \{a \in |\mathfrak{M}|^{<\omega} : R(a, n)\}; & |\mathfrak{N}_n|^{<\omega} &= \{a \in |\mathfrak{N}|^{<\omega} : S(a, n)\}. \end{aligned}$$

The **structure** \mathfrak{M} also has a ternary relation J such that $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Now there is a **sentence** θ in the **language** \mathcal{L} augmented by R, S, J , etc., saying that \leq is a discrete linear ordering with first but no last element and such that $\mathfrak{M}_n \models \alpha$, $\mathfrak{N}_n \not\models \alpha$, and for each n in the ordering, $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Using the Compactness Property, we can find a model \mathfrak{M}^* of θ in which the ordering contains a non-standard element n^* . In particular then \mathfrak{M}^* will

contain substructures \mathfrak{M}_{n^*} and \mathfrak{N}_{n^*} such that $\mathfrak{M}_{n^*} \models_L \alpha$ and $\mathfrak{N}_{n^*} \not\models_L \alpha$. But now we can define a set \mathcal{I} of pairs of k -tuples from $|\mathfrak{M}_{n^*}|$ and $|\mathfrak{N}_{n^*}|$ by putting $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathcal{I}$ if and only if $J(n^* - k, \mathbf{a}, \mathbf{b})$, where k is the length of \mathbf{a} and \mathbf{b} . Since n^* is non-standard, for each standard k we have that $n^* - k > 0$, and the set \mathcal{I} witnesses the fact that $\mathfrak{M}_{n^*} \simeq_p \mathfrak{N}_{n^*}$. But by [Theorem lin.7](#), \mathfrak{M}_{n^*} is L -equivalent to \mathfrak{N}_{n^*} , a contradiction. \square

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Bibliography