**Lemma lin.1.** Suppose \( \alpha \in L(\mathcal{L}) \), with \( \mathcal{L} \) finite, and assume also that there is an \( n \in \mathbb{N} \) such that for any two structures \( \mathfrak{M} \) and \( \mathfrak{N} \), if \( \mathfrak{M} \equiv_n \mathfrak{N} \) and \( \mathfrak{M} \models_L \alpha \) then also \( \mathfrak{N} \models_L \alpha \). Then \( \alpha \) is equivalent to a first-order sentence, i.e., there is a first-order \( \theta \) such that \( \text{Mod}_L(\alpha) = \text{Mod}_L(\theta) \).

**Proof.** Let \( n \) be such that any two \( n \)-equivalent structures \( \mathfrak{M} \) and \( \mathfrak{N} \) agree on the value assigned to \( \alpha \). Recall ??: there are only finitely many first-order sentences in a finite language that have quantifier rank no greater than \( n \), up to logical equivalence. Now, for each fixed structure \( \mathfrak{M} \) let \( \theta_{\mathfrak{M}} \) be the conjunction of all first-order sentences \( \alpha \) true in \( \mathfrak{M} \) with \( \text{qr}(\alpha) \leq n \) (this conjunction is finite), so that \( \mathfrak{M} \models \theta_{\mathfrak{M}} \) if and only if \( \mathfrak{M} \equiv_n \mathfrak{M} \). Then put \( \theta = \bigvee \{ \theta_{\mathfrak{M}} : \mathfrak{M} \models_L \alpha \} \); this disjunction is also finite (up to logical equivalence).

The conclusion \( \text{Mod}_L(\alpha) = \text{Mod}_L(\theta) \) follows. In fact, if \( \mathfrak{M} \models_L \theta \) then for some \( \mathfrak{M} \models_L \alpha \) we have \( \mathfrak{M} \models \theta_{\mathfrak{M}} \), whence also \( \mathfrak{N} \models_L \alpha \) (by the hypothesis of the lemma). Conversely, if \( \mathfrak{M} \models_L \alpha \) then \( \theta_{\mathfrak{M}} \) is a disjunct in \( \theta \), and since \( \mathfrak{M} \models \theta_{\mathfrak{M}} \), also \( \mathfrak{N} \models_L \theta \).

**Theorem lin.2 (Lindström’s Theorem).** Suppose \( \langle L, \models_L \rangle \) has the Compactness and the Löwenheim-Skolem Properties. Then \( \langle L, \models_L \rangle \leq \langle F, \models \rangle \) (so \( \langle L, \models_L \rangle \) is equivalent to first-order logic).

**Proof.** By **Lemma lin.1**, it suffices to show that for any \( \alpha \in L(\mathcal{L}) \), with \( \mathcal{L} \) finite, there is \( n \in \mathbb{N} \) such that for any two structures \( \mathfrak{M} \) and \( \mathfrak{N} \), if \( \mathfrak{M} \equiv_n \mathfrak{N} \) then \( \mathfrak{M} \) and \( \mathfrak{N} \) agree on \( \alpha \). For then \( \alpha \) is equivalent to a first-order sentence, from which \( \langle L, \models_L \rangle \leq \langle F, \models \rangle \) follows. Since we are working in a finite, purely relational language, by ?? we can replace the statement that \( \mathfrak{M} \equiv_n \mathfrak{N} \) by the corresponding algebraic statement that \( I_n(\emptyset, \emptyset) \).

Given \( \alpha \), suppose towards a contradiction that for each \( n \) there are structures \( \mathfrak{M}_n \) and \( \mathfrak{N}_n \) such that \( I_n(\emptyset, \emptyset) \), but (say) \( \mathfrak{M}_n \models_L \alpha \) whereas \( \mathfrak{N}_n \not\models_L \alpha \). By the Isomorphism Property we can assume that all the \( \mathfrak{M}_n \)'s interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic sentences in the language, we may also assume that they satisfy the same atomic sentences (we can take a subsequence of the \( \mathfrak{M} \)’s otherwise). Let \( \mathfrak{M} \) be the union of all the \( \mathfrak{M}_n \)'s, i.e., the unique minimal structure having each \( \mathfrak{M}_n \) as a substructure. As in the proof of ??, let \( \mathfrak{M}^+ \) be the extension of \( \mathfrak{M} \) with domain \( \mathfrak{M} \cup \mathfrak{M}^{<\omega} \), in the expanded language comprising the concatenation predicates \( P \) and \( Q \).

Similarly, define \( \mathfrak{N}_n, \mathfrak{N} \) and \( \mathfrak{N}^+ \). Now let \( \mathfrak{M} \) be the structure whose domain comprises the domains of \( \mathfrak{M}^+ \) and \( \mathfrak{N}^+ \) as well as the natural numbers \( \mathbb{N} \) along with their natural ordering \( \leq \), in the language with extra predicates representing the domains \( |\mathfrak{M}|, |\mathfrak{M}^{<\omega}| \) and \( |\mathfrak{M}|^{<\omega} \) as well as predicates coding the
The structure $\mathcal{M}$ also has a ternary relation $J$ such that $J(n, a, b)$ holds if and only if $I_n(a, b)$.

Now there is a sentence $\theta$ in the language $L$ augmented by $R, S, J$, etc., saying that $\leq$ is a discrete linear ordering with first but no last element and such that $\mathcal{M}_n \models \alpha$, $\mathcal{N}_n \not\models \alpha$, and for each $n$ in the ordering, $J(n, a, b)$ holds if and only if $I_n(a, b)$.

Using the Compactness Property, we can find a model $\mathcal{M}^*$ of $\theta$ in which the ordering contains a non-standard element $n^*$. In particular then $\mathcal{M}^*$ will contain substructures $\mathcal{M}_{n^*}$ and $\mathcal{N}_{n^*}$ such that $\mathcal{M}_{n^*} \models_L \alpha$ and $\mathcal{N}_{n^*} \not\models_L \alpha$. But now we can define a set $\mathcal{I}$ of pairs of $k$-tuples from $|\mathcal{M}_{n^*}|$ and $|\mathcal{N}_{n^*}|$ by putting $(a, b) \in \mathcal{I}$ if and only if $J(n^* - k, a, b)$, where $k$ is the length of $a$ and $b$. Since $n^*$ is non-standard, for each standard $k$ we have that $n^* - k > 0$, and the set $\mathcal{I}$ witnesses the fact that $\mathcal{M}_{n^*} \simeq_p \mathcal{N}_{n^*}$. But by ??, $\mathcal{M}_{n^*}$ is $L$-equivalent to $\mathcal{N}_{n^*}$, a contradiction.

\[\square\]

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Bibliography