lin.1 Lindström's Theorem

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mod:lin:prf:lem:lindstrom **Lemma lin.1.** *Suppose $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, and assume also that there is an $n \in \mathbb{N}$ such that for any two structures $\mathfrak{M}$ and $\mathfrak{N}$, if $\mathfrak{M} \equiv_n \mathfrak{N}$ and $\mathfrak{M} \models_L \alpha$ then also $\mathfrak{N} \models_L \alpha$. Then α is equivalent to a first-order sentence, i.e., there is a first-order θ such that $\mathrm{Mod}_L(\alpha) = \mathrm{Mod}_L(\theta)$.*

Proof. Let n be such that any two n-equivalent structures $\mathfrak{M}$ and $\mathfrak{N}$ agree on the value assigned to α. Recall **??**: there are only finitely many first-order sentences in a finite language that have quantifier rank no greater than n, up to logical equivalence. Now, for each fixed structure $\mathfrak{M}$ let $\theta_{\mathfrak{M}}$ be the conjunction of all first-order sentences α true in $\mathfrak{M}$ with $\mathrm{qr}(\alpha) \leq n$ (this conjunction is finite), so that $\mathfrak{N} \models \theta_{\mathfrak{M}}$ if and only if $\mathfrak{N} \equiv_n \mathfrak{M}$. Then put $\theta = \bigvee\{\theta_{\mathfrak{M}} : \mathfrak{M} \models_L \alpha\}$; this disjunction is also finite (up to logical equivalence).

The conclusion $\mathrm{Mod}_L(\alpha) = \mathrm{Mod}_L(\theta)$ follows. In fact, if $\mathfrak{N} \models_L \theta$ then for some $\mathfrak{M} \models_L \alpha$ we have $\mathfrak{N} \models \theta_{\mathfrak{M}}$, whence also $\mathfrak{N} \models_L \alpha$ (by the hypothesis of the lemma). Conversely, if $\mathfrak{N} \models_L \alpha$ then $\theta_{\mathfrak{N}}$ is a disjunct in θ, and since $\mathfrak{N} \models \theta_{\mathfrak{N}}$, also $\mathfrak{N} \models_L \theta$. □

mod:lin:prf:thm:lindstrom **Theorem lin.2** (Lindström's Theorem)**.** *Suppose $\langle L, \models_L\rangle$ has the Compactness and the Löwenheim-Skolem Properties. Then $\langle L, \models_L\rangle \leq \langle F, \models\rangle$ (so $\langle L, \models_L\rangle$ is equivalent to first-order logic).*

Proof. By Lemma lin.1, it suffices to show that for any $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, there is $n \in \mathbb{N}$ such that for any two structures $\mathfrak{M}$ and $\mathfrak{N}$: if $\mathfrak{M} \equiv_n \mathfrak{N}$ then $\mathfrak{M}$ and $\mathfrak{N}$ agree on α. For then α is equivalent to a first-order sentence, from which $\langle L, \models_L\rangle \leq \langle F, \models\rangle$ follows. Since we are working in a finite, purely relational language, by **??** we can replace the statement that $\mathfrak{M} \equiv_n \mathfrak{N}$ by the corresponding algebraic statement that $I_n(\emptyset, \emptyset)$.

Given α, suppose towards a contradiction that for each n there are structures $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that $I_n(\emptyset, \emptyset)$, but (say) $\mathfrak{M}_n \models_L \alpha$ whereas $\mathfrak{N}_n \not\models_L \alpha$. By the Isomorphism Property we can assume that all the $\mathfrak{M}_n$'s interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic sentences in the language, we may also assume that they satisfy the same atomic sentences (we can take a subsequence of the $\mathfrak{M}$'s otherwise). Let $\mathfrak{M}$ be the union of all the $\mathfrak{M}_n$'s, i.e., the unique minimal structure having each $\mathfrak{M}_n$ as a substructure. As in the proof of **??**, let $\mathfrak{M}^*$ be the extension of $\mathfrak{M}$ with domain $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, in the expanded language comprising the concatenation predicates P and Q.

Similarly, define $\mathfrak{N}_n$, $\mathfrak{N}$ and $\mathfrak{N}^*$. Now let $\mathfrak{M}$ be the structure whose domain comprises the domains of $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as well as the natural numbers $\mathbb{N}$ along with their natural ordering $\leq$, in the language with extra predicates representing the domains $|\mathfrak{M}|$, $|\mathfrak{N}|$, $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ as well as predicates coding the domains of $\mathfrak{M}_n$ and $\mathfrak{N}_n$ in the sense that:

$$|\mathfrak{M}_n| = \{a \in |\mathfrak{M}| : R(a,n)\}; \qquad |\mathfrak{N}_n| = \{a \in |\mathfrak{N}| : S(a,n)\};$$
$$|\mathfrak{M}|_n^{<\omega} = \{a \in |\mathfrak{M}|^{<\omega} : R(a,n)\}; \qquad |\mathfrak{N}|_n^{<\omega} = \{a \in |\mathfrak{N}|^{<\omega} : S(a,n)\}.$$

The structure $\mathfrak{M}$ also has a ternary relation J such that $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Now there is a sentence θ in the language $\mathcal{L}$ augmented by R, S, J, etc., saying that $\leq$ is a discrete linear ordering with first but no last element and such that $\mathfrak{M}_n \models \alpha$, $\mathfrak{N}_n \not\models \alpha$, and for each n in the ordering, $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Using the Compactness Property, we can find a model $\mathfrak{M}^*$ of θ in which the ordering contains a non-standard element n^*. In particular then $\mathfrak{M}^*$ will contain substructures $\mathfrak{M}_{n^*}$ and $\mathfrak{N}_{n^*}$ such that $\mathfrak{M}_{n^*} \models_L \alpha$ and $\mathfrak{N}_{n^*} \not\models_L \alpha$. But now we can define a set $\mathcal{I}$ of pairs of k-tuples from $|\mathfrak{M}_{n^*}|$ and $|\mathfrak{N}_{n^*}|$ by putting $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathcal{I}$ if and only if $J(n^* - k, \mathbf{a}, \mathbf{b})$, where k is the length of $\mathbf{a}$ and $\mathbf{b}$. Since n^* is non-standard, for each standard k we have that $n^* - k > 0$, and the set $\mathcal{I}$ witnesses the fact that $\mathfrak{M}_{n^*} \simeq_p \mathfrak{N}_{n^*}$. But by **??**, $\mathfrak{M}_{n^*}$ is L-equivalent to $\mathfrak{N}_{n^*}$, a contradiction. □

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Bibliography