Lemma lin.1. Suppose $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, and assume also that there is an $n \in \mathbb{N}$ such that for any two structures $\mathcal{M}$ and $\mathcal{N}$, if $\mathcal{M} \equiv_n \mathcal{N}$ and $\mathcal{M} \models_L \alpha$ then also $\mathcal{N} \models_L \alpha$. Then $\alpha$ is equivalent to a first-order sentence, i.e., there is a first-order $\theta$ such that $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$.

Proof. Let $n$ be such that any two $n$-equivalent structures $\mathcal{M}$ and $\mathcal{N}$ agree on the value assigned to $\alpha$. Recall ??: there are only finitely many first-order sentences in a finite language that have quantifier rank no greater than $n$, up to logical equivalence. Now, for each fixed structure $\mathcal{M}$ let $\theta_{\mathcal{M}}$ be the conjunction of all first-order sentences $\alpha$ true in $\mathcal{M}$ with $qr(\alpha) \leq n$ (this conjunction is finite), so that $\mathcal{N} \models \theta_{\mathcal{M}}$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$. Then put $\theta = \bigvee \{ \theta_{\mathcal{M}} : \mathcal{M} \models_L \alpha \}$; this disjunction is also finite (up to logical equivalence).

The conclusion $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$ follows. In fact, if $\mathcal{N} \models_L \theta$ then for some $\mathcal{M} \models_L \alpha$ we have $\mathcal{N} \models \theta_{\mathcal{M}}$, whence also $\mathcal{N} \models_L \alpha$ (by the hypothesis of the lemma). Conversely, if $\mathcal{N} \models_L \alpha$ then $\theta_{\mathcal{M}}$ is a disjunct in $\theta$, and since $\mathcal{N} \models \theta_{\mathcal{M}}$, also $\mathcal{N} \models_L \theta$. \hfill $\square$

Theorem lin.2 (Lindström’s Theorem). Suppose $\langle L, \models_L \rangle$ has the Compactness and the Löwenheim-Skolem Properties. Then $\langle L, \models_L \rangle \leq \langle F, \models \rangle$ (so $\langle L, \models_L \rangle$ is equivalent to first-order logic).

Proof. By Lemma lin.1, it suffices to show that for any $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, there is $n \in \mathbb{N}$ such that for any two structures $\mathcal{M}$ and $\mathcal{N}$, if $\mathcal{M} \equiv_n \mathcal{N}$ then $\mathcal{M}$ and $\mathcal{N}$ agree on $\alpha$. For then $\alpha$ is equivalent to a first-order sentence, from which $\langle L, \models_L \rangle \leq \langle F, \models \rangle$ follows. Since we are working in a finite, purely relational language, by ?? we can replace the statement that $\mathcal{M} \equiv_n \mathcal{N}$ by the corresponding algebraic statement that $I_n(\emptyset, \emptyset)$.

Given $\alpha$, suppose towards a contradiction that for each $n$ there are structures $\mathcal{M}_n$ and $\mathcal{N}_n$ such that $I_n(\emptyset, \emptyset)$, but (say) $\mathcal{M}_n \models_L \alpha$ whereas $\mathcal{N}_n \not\models_L \alpha$. By the Isomorphism Property we can assume that all the $\mathcal{M}_n$’s interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic sentences in the language, we may also assume that they satisfy the same atomic sentences (we can take a subsequence of the $\mathcal{M}_n$’s otherwise). Let $\mathcal{M}$ be the union of all the $\mathcal{M}_n$’s, i.e., the unique minimal structure having each $\mathcal{M}_n$ as a substructure. As in the proof of ??, let $\mathcal{M}^*$ be the extension of $\mathcal{M}$ with domain $|\mathcal{M}| \cup |\mathcal{M}|^\omega$, in the expanded language comprising the concatenation predicates $P$ and $Q$.

Similarly, define $\mathcal{N}_n$, $\mathcal{N}$ and $\mathcal{N}^*$. Now let $\mathcal{M}$ be the structure whose domain comprises the domains of $\mathcal{M}^*$ and $\mathcal{N}^*$ as well as the natural numbers $\mathbb{N}$ along with their natural ordering $\leq$, in the language with extra predicates representing the domains $|\mathcal{M}|$, $|\mathcal{N}|$, $|\mathcal{M}|^\omega$ and $|\mathcal{N}|^\omega$ as well as predicates coding the
domains of $\mathcal{M}_n$ and $\mathcal{N}_n$ in the sense that:

$$|\mathcal{M}_n| = \{a \in |\mathcal{M}| : R(a, n)\}; \quad |\mathcal{N}_n| = \{a \in |\mathcal{N}| : S(a, n)\};$$

$$|\mathcal{M}|^{<\omega}_n = \{a \in |\mathcal{M}|^{<\omega} : R(a, n)\}; \quad |\mathcal{N}|^{<\omega}_n = \{a \in |\mathcal{N}|^{<\omega} : S(a, n)\}.$$

The structure $\mathcal{M}$ also has a ternary relation $J$ such that $J(n, a, b)$ holds if and only if $I_n(a, b)$.

Now there is a sentence $\theta$ in the language $L$ augmented by $R$, $S$, $J$, etc., saying that $\leq$ is a discrete linear ordering with first but no last element and such that $\mathcal{M}_n \models = \alpha$, $\mathcal{N}_n \not\models = \alpha$, and for each $n$ in the ordering, $J(n, a, b)$ holds if and only if $I_n(a, b)$.

Using the Compactness Property, we can find a model $\mathcal{M}^*$ of $\theta$ in which the ordering contains a non-standard element $n^*$. In particular then $\mathcal{M}^*$ will contain substructures $\mathcal{M}_{n^*}$ and $\mathcal{N}_{n^*}$ such that $\mathcal{M}_{n^*} \models_L = \alpha$ and $\mathcal{N}_{n^*} \not\models_L = \alpha$. But now we can define a set $I$ of pairs of $k$-tuples from $|\mathcal{M}_{n^*}|$ and $|\mathcal{N}_{n^*}|$ by putting $(a, b) \in I$ if and only if $J(n^* - k, a, b)$, where $k$ is the length of $a$ and $b$. Since $n^*$ is non-standard, for each standard $k$ we have that $n^* - k > 0$, and the set $I$ witnesses the fact that $\mathcal{M}_{n^*} \simeq_p \mathcal{N}_{n^*}$. But by $??$, $\mathcal{M}_{n^*}$ is $L$-equivalent to $\mathcal{N}_{n^*}$, a contradiction.

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Bibliography