

lin.1 Lindström's Theorem

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lem:lindstrom* **Lemma lin.1.** *Suppose $\alpha \in L(\mathcal{L})$, with \mathcal{L} finite, and assume also that there is an $n \in \mathbb{N}$ such that for any two structures \mathfrak{M} and \mathfrak{N} , if $\mathfrak{M} \equiv_n \mathfrak{N}$ and $\mathfrak{M} \models_L \alpha$ then also $\mathfrak{N} \models_L \alpha$. Then α is equivalent to a first-order sentence, i.e., there is a first-order θ such that $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$.*

Proof. Let n be such that any two n -equivalent structures \mathfrak{M} and \mathfrak{N} agree on the value assigned to α . Recall ???: there are only finitely many first-order sentences in a finite language that have quantifier rank no greater than n , up to logical equivalence. Now, for each fixed structure \mathfrak{M} let $\theta_{\mathfrak{M}}$ be the conjunction of all first-order sentences α true in \mathfrak{M} with $\text{qr}(\alpha) \leq n$ (this conjunction is finite), so that $\mathfrak{N} \models \theta_{\mathfrak{M}}$ if and only if $\mathfrak{N} \equiv_n \mathfrak{M}$. Then put $\theta = \bigvee \{\theta_{\mathfrak{M}} : \mathfrak{M} \models_L \alpha\}$; this disjunction is also finite (up to logical equivalence).

The conclusion $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$ follows. In fact, if $\mathfrak{N} \models_L \theta$ then for some $\mathfrak{M} \models_L \alpha$ we have $\mathfrak{N} \models \theta_{\mathfrak{M}}$, whence also $\mathfrak{N} \models_L \alpha$ (by the hypothesis of the lemma). Conversely, if $\mathfrak{N} \models_L \alpha$ then $\theta_{\mathfrak{N}}$ is a disjunct in θ , and since $\mathfrak{N} \models \theta_{\mathfrak{N}}$, also $\mathfrak{N} \models_L \theta$. \square

*mod:lin:prf:
thm:lindstrom* **Theorem lin.2** (Lindström's Theorem). *Suppose $\langle L, \models_L \rangle$ has the Compactness and the Löwenheim-Skolem Properties. Then $\langle L, \models_L \rangle \leq \langle F, \models \rangle$ (so $\langle L, \models_L \rangle$ is equivalent to first-order logic).*

Proof. By Lemma lin.1, it suffices to show that for any $\alpha \in L(\mathcal{L})$, with \mathcal{L} finite, there is $n \in \mathbb{N}$ such that for any two structures \mathfrak{M} and \mathfrak{N} : if $\mathfrak{M} \equiv_n \mathfrak{N}$ then \mathfrak{M} and \mathfrak{N} agree on α . For then α is equivalent to a first-order sentence, from which $\langle L, \models_L \rangle \leq \langle F, \models \rangle$ follows. Since we are working in a finite, purely relational language, by ??? we can replace the statement that $\mathfrak{M} \equiv_n \mathfrak{N}$ by the corresponding algebraic statement that $I_n(\emptyset, \emptyset)$.

Given α , suppose towards a contradiction that for each n there are structures \mathfrak{M}_n and \mathfrak{N}_n such that $I_n(\emptyset, \emptyset)$, but (say) $\mathfrak{M}_n \models_L \alpha$ whereas $\mathfrak{N}_n \not\models_L \alpha$. By the Isomorphism Property we can assume that all the \mathfrak{M}_n 's interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic sentences in the language, we may also assume that they satisfy the same atomic sentences (we can take a subsequence of the \mathfrak{M} 's otherwise). Let \mathfrak{M} be the union of all the \mathfrak{M}_n 's, i.e., the unique minimal structure having each \mathfrak{M}_n as a substructure. As in the proof of ???, let \mathfrak{M}^* be the extension of \mathfrak{M} with domain $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, in the expanded language comprising the concatenation predicates P and Q .

Similarly, define \mathfrak{N}_n , \mathfrak{N} and \mathfrak{N}^* . Now let \mathfrak{M} be the structure whose domain comprises the domains of \mathfrak{M}^* and \mathfrak{N}^* as well as the natural numbers \mathbb{N} along with their natural ordering \leq , in the language with extra predicates representing the domains $|\mathfrak{M}|$, $|\mathfrak{N}|$, $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ as well as predicates coding the

domains of \mathfrak{M}_n and \mathfrak{N}_n in the sense that:

$$\begin{aligned} |\mathfrak{M}_n| &= \{a \in |\mathfrak{M}| : R(a, n)\}; & |\mathfrak{N}_n| &= \{a \in |\mathfrak{N}| : S(a, n)\}; \\ |\mathfrak{M}_n|^{<\omega} &= \{a \in |\mathfrak{M}|^{<\omega} : R(a, n)\}; & |\mathfrak{N}_n|^{<\omega} &= \{a \in |\mathfrak{N}|^{<\omega} : S(a, n)\}. \end{aligned}$$

The **structure** \mathfrak{M} also has a ternary relation J such that $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Now there is a **sentence** θ in the **language** \mathcal{L} augmented by R, S, J , etc., saying that \leq is a discrete linear ordering with first but no last element and such that $\mathfrak{M}_n \models \alpha$, $\mathfrak{N}_n \not\models \alpha$, and for each n in the ordering, $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Using the Compactness Property, we can find a model \mathfrak{M}^* of θ in which the ordering contains a non-standard element n^* . In particular then \mathfrak{M}^* will contain **substructures** \mathfrak{M}_{n^*} and \mathfrak{N}_{n^*} such that $\mathfrak{M}_{n^*} \models_L \alpha$ and $\mathfrak{N}_{n^*} \not\models_L \alpha$. But now we can define a set \mathcal{I} of pairs of k -tuples from $|\mathfrak{M}_{n^*}|$ and $|\mathfrak{N}_{n^*}|$ by putting $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathcal{I}$ if and only if $J(n^* - k, \mathbf{a}, \mathbf{b})$, where k is the length of \mathbf{a} and \mathbf{b} . Since n^* is non-standard, for each standard k we have that $n^* - k > 0$, and the set \mathcal{I} witnesses the fact that $\mathfrak{M}_{n^*} \simeq_p \mathfrak{N}_{n^*}$. But by ??, \mathfrak{M}_{n^*} is L -equivalent to \mathfrak{N}_{n^*} , a contradiction. \square

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Bibliography