Lemma lin.1. Suppose $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, and assume also that there is an $n \in \mathbb{N}$ such that for any two structures $\mathfrak{M}$ and $\mathfrak{N}$, if $\mathfrak{M} \equiv_n \mathfrak{N}$ and $\mathfrak{M} \models L \alpha$ then also $\mathfrak{N} \models L \alpha$. Then $\alpha$ is equivalent to a first-order sentence, i.e., there is a first-order $\theta$ such that $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$.

Proof. Let $n$ be such that any two $n$-equivalent structures $\mathfrak{M}$ and $\mathfrak{N}$ agree on the value assigned to $\alpha$. Recall ??: there are only finitely many first-order sentences in a finite language that have quantifier rank no greater than $n$, up to logical equivalence. Now, for each fixed structure $\mathfrak{M}$ let $\theta_{\mathfrak{M}}$ be the conjunction of all first-order sentences $\alpha$ true in $\mathfrak{M}$ with $qr(\alpha) \leq n$ (this conjunction is finite), so that $\mathfrak{M} \models \theta_{\mathfrak{M}}$ if and only if $\mathfrak{M} \equiv_n \mathfrak{M}$. Then put $\theta = \bigvee \{ \theta_{\mathfrak{M}} : \mathfrak{M} \models L \alpha \}$; this disjunction is also finite (up to logical equivalence).

The conclusion $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$ follows. In fact, if $\mathfrak{M} \models L \theta$ then for some $\mathfrak{M} \models L \alpha$ we have $\mathfrak{M} \models \theta_{\mathfrak{M}}$, whence also $\mathfrak{M} \models L \alpha$ (by the hypothesis of the lemma). Conversely, if $\mathfrak{M} \models L \alpha$ then $\theta_{\mathfrak{M}}$ is a disjunct in $\theta$, and since $\mathfrak{M} \models \theta_{\mathfrak{M}}$, also $\mathfrak{M} \models L \theta$. \qed

Theorem lin.2 (Lindström’s Theorem). Suppose $(L, \models L)$ has the Compactness and the Löwenheim-Skolem Properties. Then $(L, \models L) \leq (F, \models)$ (so $(L, \models L)$ is equivalent to first-order logic).

Proof. By Lemma lin.1, it suffices to show that for any $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, there is $n \in \mathbb{N}$ such that for any two structures $\mathfrak{M}$ and $\mathfrak{N}$ if $\mathfrak{M} \equiv_n \mathfrak{N}$ then $\mathfrak{M}$ and $\mathfrak{N}$ agree on $\alpha$. For then $\alpha$ is equivalent to a first-order sentence, from which $(L, \models L) \leq (F, \models)$ follows. Since we are working in a finite, purely relational language, by ?? we can replace the statement that $\mathfrak{M} \equiv_n \mathfrak{N}$ by the corresponding algebraic statement that $I_n(\emptyset, \emptyset)$.

Given $\alpha$, suppose towards a contradiction that for each $n$ there are structures $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that $I_n(\emptyset, \emptyset)$, but (say) $\mathfrak{M}_n \models L \alpha$ whereas $\mathfrak{N}_n \not\models L \alpha$. By the Isomorphism Property we can assume that all the $\mathfrak{M}_n$’s interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic sentences in the language, we may also assume that they satisfy the same atomic sentences (we can take a subsequence of the $\mathfrak{M}_n$’s otherwise). Let $\mathfrak{M}$ be the union of all the $\mathfrak{M}_n$’s, i.e., the unique minimal structure having each $\mathfrak{M}_n$ as a substructure. As in the proof of ??, let $\mathfrak{M}^*$ be the extension of $\mathfrak{M}$ with domain $|\mathfrak{M}|\cup |\mathfrak{M}|^{<\omega}$, in the expanded language comprising the concatenation predicates $P$ and $Q$.

Similarly, define $\mathfrak{N}_n$, $\mathfrak{N}$ and $\mathfrak{N}^*$. Now let $\mathfrak{M}$ be the structure whose domain comprises the domains of $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as well as the natural numbers $\mathbb{N}$ along with their natural ordering $<\omega$, in the language with extra predicates representing the domains $|\mathfrak{M}|$, $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{M}|^{<\omega}$ as well as predicates coding the
domains of $\mathcal{M}_n$ and $\mathcal{N}_n$ in the sense that:

$$|\mathcal{M}_n| = \{a \in |\mathcal{M}| : R(a, n)\}; \quad |\mathcal{N}_n| = \{a \in |\mathcal{N}| : S(a, n)\};$$

$$|\mathcal{M}|^{<\omega}_n = \{a \in |\mathcal{M}|^{<\omega} : R(a, n)\}; \quad |\mathcal{N}|^{<\omega}_n = \{a \in |\mathcal{N}|^{<\omega} : S(a, n)\}.$$  

The structure $\mathcal{M}$ also has a ternary relation $J$ such that $J(n, a, b)$ holds if and only if $I_n(a, b)$.

Now there is a sentence $\theta$ in the language $\mathcal{L}$ augmented by $R$, $S$, $J$, etc., saying that $\leq$ is a discrete linear ordering with first but no last element and such that $\mathcal{M}_n \models \alpha$, $\mathcal{N}_n \not\models \alpha$, and for each $n$ in the ordering, $J(n, a, b)$ holds if and only if $I_n(a, b)$.

Using the Compactness Property, we can find a model $\mathcal{M}^*$ of $\theta$ in which the ordering contains a non-standard element $n^*$. In particular then $\mathcal{M}^*$ will contain substructures $\mathcal{M}_n^*$ and $\mathcal{N}_n^*$ such that $\mathcal{M}_n^* \models_L \alpha$ and $\mathcal{N}_n^* \not\models_L \alpha$. But now we can define a set $I$ of pairs of $k$-tuples from $|\mathcal{M}_n^*|$ and $|\mathcal{N}_n^*|$ by putting $(a, b) \in I$ if and only if $J(n^* - k, a, b)$, where $k$ is the length of $a$ and $b$. Since $n^*$ is non-standard, for each standard $k$ we have that $n^* - k > 0$, and the set $I$ witnesses the fact that $\mathcal{M}_n^* \simeq_p \mathcal{N}_n^*$. But by ??, $\mathcal{M}_n^*$ is $L$-equivalent to $\mathcal{N}_n^*$, a contradiction.  

\[\square\]

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Bibliography