Definition lin.1. An abstract logic is a pair \( \langle L, \models_L \rangle \), where \( L \) is a function that assigns to each language \( L \) a set \( L(L) \) of sentences, and \( \models_L \) is a relation between structures for the language \( L \) and elements of \( L(L) \). In particular, \( \langle F, \models \rangle \) is ordinary first-order logic, i.e., \( F \) is the function assigning to the language \( L \) the set of first-order sentences built from the constants in \( L \), and \( \models \) is the satisfaction relation of first-order logic.

Notice that we are still employing the same notion of structure for a given language as for first-order logic, but we do not presuppose that sentences are build up from the basic symbols in \( L \) in the usual way, nor that the relation \( \models_L \) is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where sentences in \( \langle L, \models_L \rangle \) contain infinitely long conjunctions or disjunction, or quantifiers other than \( \exists \) and \( \forall \) (e.g., “there are infinitely many \( x \) such that . . . ”), or perhaps infinitely long quantifier prefixes. To emphasize that “sentences” in \( L(L) \) need not be ordinary sentences of first-order logic, in this chapter we use variables \( \alpha, \beta, \ldots \) to range over them, and reserve \( \varphi, \psi, \ldots \) for ordinary first-order formulas.

Definition lin.2. Let \( \text{Mod}^L(\alpha) \) denote the class \( \{ M : M \models_L \alpha \} \). If the language needs to be made explicit, we write \( \text{Mod}^L(\alpha) \). Two structures \( M \) and \( N \) for \( L \) are elementarily equivalent in \( \langle L, \models_L \rangle \), written \( M \equiv_L N \), if the same sentences from \( L(L) \) are true in each.

Definition lin.3. An abstract logic \( \langle L, \models_L \rangle \) for the language \( L \) is normal if it satisfies the following properties:

1. \((L\text{-Monotony})\) For languages \( L \) and \( L' \), if \( L \subseteq L' \), then \( L(L) \subseteq L(L') \).

2. \((\text{Expansion Property})\) For each \( \alpha \in L(L) \) there is a finite subset \( L' \) of \( L \) such that the relation \( M \models_L \alpha \) depends only on the reduct of \( M \) to \( L' \); i.e., if \( M \) and \( N \) have the same reduct to \( L' \) then \( M \models_L \alpha \) if and only if \( N \models_L \alpha \).

3. \((\text{Isomorphism Property})\) If \( M \models_L \alpha \) and \( M \cong N \) then also \( N \models_L \alpha \).

4. \((\text{Renaming Property})\) The relation \( \models_L \) is preserved under renaming: if the language \( L' \) is obtained from \( L \) by replacing each symbol \( P \) by a symbol \( P' \) of the same arity and each constant \( c \) by a distinct constant \( c' \), then for each structure \( M \) and sentence \( \alpha \), \( M \models_L \alpha \) if and only if \( M' \models_L \alpha' \), where \( M' \) is the \( L' \)-structure corresponding to \( L \) and \( \alpha' \in L(L') \).

5. \((\text{Boolean Property})\) The abstract logic \( \langle L, \models_L \rangle \) is closed under the Boolean connectives in the sense that for each \( \alpha \in L(L) \) there is a \( \beta \in L(L) \) such that \( M \models_L \beta \) if and only if \( M \models_L \alpha \), and for each \( \alpha \) and \( \beta \) there is a \( \gamma \).
such that $\text{Mod}_L(\gamma) = \text{Mod}_L(\alpha) \cap \text{Mod}_L(\beta)$. Similarly for atomic formulas and the other connectives.

6. (*Quantifier Property*) For each constant $c$ in $\mathcal{L}$ and $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that

$$\text{Mod}_L'(\beta) = \{ \mathcal{M} : (\mathcal{M}, a) \in \text{Mod}_L(\alpha) \text{ for some } a \in |\mathcal{M}| \},$$

where $\mathcal{L}' = \mathcal{L} \setminus \{ c \}$ and $(\mathcal{M}, a)$ is the expansion of $\mathcal{M}$ to $\mathcal{L}$ assigning $a$ to $c$.

7. (*Relativization Property*) Given a sentence $\alpha \in L(\mathcal{L})$ and symbols $R, c_1, \ldots, c_n$ not in $\mathcal{L}$, there is a sentence $\beta \in L(\mathcal{L} \cup \{ R, c_1, \ldots, c_n \})$ called the relativization of $\alpha$ to $R(x, c_1, \ldots, c_n)$, such that for each structure $\mathcal{M}$:

$$(\mathcal{M}, X, b_1, \ldots, b_n) \models L \beta$$

if and only if $\mathcal{M} \models L \alpha$, where $\mathcal{M}$ is the substructure of $\mathcal{M}$ with domain $|\mathcal{M}| = \{ a \in |\mathcal{M}| : R^{\mathcal{M}}(a, b_1, \ldots, b_n) \}$ (see ??), and $(\mathcal{M}, X, b_1, \ldots, b_n)$ is the expansion of $\mathcal{M}$ interpreting $R, c_1, \ldots, c_n$ by $X, b_1, \ldots, b_n$, respectively (with $X \subseteq M^{n+1}$).

**Definition lin.4.** Given two abstract logics $\langle L_1, \models_L \rangle$ and $\langle L_2, \models_L \rangle$ we say that the latter is at least as expressive as the former, written $\langle L_1, \models_L \rangle \leq \langle L_2, \models_L \rangle$, if for each language $\mathcal{L}$ and sentence $\alpha \in L_1(\mathcal{L})$ there is a sentence $\beta \in L_2(\mathcal{L})$ such that $\text{Mod}_L(\alpha) = \text{Mod}_L(\beta)$. The logics $\langle L_1, \models_L \rangle$ and $\langle L_2, \models_L \rangle$ are equivalent if $\langle L_1, \models_L \rangle \leq \langle L_2, \models_L \rangle$ and $\langle L_2, \models_L \rangle \leq \langle L_1, \models_L \rangle$.

**Remark 1.** First-order logic, i.e., the abstract logic $\langle F, \models \rangle$, is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a sentence $\alpha$ to $R(x, c_1, \ldots, c_n)$ is obtained by replacing each subformula $\forall x \beta$ by $\forall x (R(x, c_1, \ldots, c_n) \rightarrow \beta)$. Moreover, if $\langle L, \models_L \rangle$ is normal, then $\langle F, \models \rangle \leq \langle L, \models_L \rangle$, as can be shown by induction on first-order formulas. Accordingly, with no loss in generality, we can assume that every first-order sentence belongs to every normal logic.

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**Bibliography**

2