

## lin.1 Abstract Logics

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**Definition lin.1.** An *abstract logic* is a pair  $\langle L, \models_L \rangle$ , where  $L$  is a function that assigns to each **language**  $\mathcal{L}$  a set  $L(\mathcal{L})$  of **sentences**, and  $\models_L$  is a relation between **structures** for the **language**  $\mathcal{L}$  and **elements** of  $L(\mathcal{L})$ . In particular,  $\langle F, \models \rangle$  is ordinary first-order logic, i.e.,  $F$  is the function assigning to the **language**  $\mathcal{L}$  the set of first-order **sentences** built from the constants in  $\mathcal{L}$ , and  $\models$  is the satisfaction relation of first-order logic.

Notice that we are still employing the same notion of **structure** for a given **language** as for first-order logic, but we do not presuppose that **sentences** are built up from the basic symbols in  $\mathcal{L}$  in the usual way, nor that the relation  $\models_L$  is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where **sentences** in  $\langle L, \models_L \rangle$  contain infinitely long conjunctions or disjunction, or quantifiers other than  $\exists$  and  $\forall$  (e.g., “there are infinitely many  $x$  such that ...”), or perhaps infinitely long quantifier prefixes. To emphasize that “**sentences**” in  $L(\mathcal{L})$  need not be ordinary **sentences** of first-order logic, in this chapter we use **variables**  $\alpha, \beta, \dots$  to range over them, and reserve  $\varphi, \psi, \dots$  for ordinary first-order **formulas**.

**Definition lin.2.** Let  $\text{Mod}_L(\alpha)$  denote the class  $\{\mathfrak{M} : \mathfrak{M} \models_L \alpha\}$ . If the **language** needs to be made explicit, we write  $\text{Mod}_L^{\mathcal{L}}(\alpha)$ . Two **structures**  $\mathfrak{M}$  and  $\mathfrak{N}$  for  $\mathcal{L}$  are *elementarily equivalent* in  $\langle L, \models_L \rangle$ , written  $\mathfrak{M} \equiv_L \mathfrak{N}$ , if the same **sentences** from  $L(\mathcal{L})$  are true in each.

**Definition lin.3.** An abstract logic  $\langle L, \models_L \rangle$  for the **language**  $\mathcal{L}$  is *normal* if it satisfies the following properties:

1. (*L-Monotony*) For **languages**  $\mathcal{L}$  and  $\mathcal{L}'$ , if  $\mathcal{L} \subseteq \mathcal{L}'$ , then  $L(\mathcal{L}) \subseteq L(\mathcal{L}')$ .
2. (*Expansion Property*) For each  $\alpha \in L(\mathcal{L})$  there is a *finite* subset  $\mathcal{L}'$  of  $\mathcal{L}$  such that the relation  $\mathfrak{M} \models_L \alpha$  depends only on the reduct of  $\mathfrak{M}$  to  $\mathcal{L}'$ ; i.e., if  $\mathfrak{M}$  and  $\mathfrak{N}$  have the same reduct to  $\mathcal{L}'$  then  $\mathfrak{M} \models_L \alpha$  if and only if  $\mathfrak{N} \models_L \alpha$ .
3. (*Isomorphism Property*) If  $\mathfrak{M} \models_L \alpha$  and  $\mathfrak{M} \simeq \mathfrak{N}$  then also  $\mathfrak{N} \models_L \alpha$ .
4. (*Renaming Property*) The relation  $\models_L$  is preserved under renaming: if the **language**  $\mathcal{L}'$  is obtained from  $\mathcal{L}$  by replacing each symbol  $P$  by a symbol  $P'$  of the same arity and each constant  $c$  by a distinct constant  $c'$ , then for each **structure**  $\mathfrak{M}$  and **sentence**  $\alpha$ ,  $\mathfrak{M} \models_L \alpha$  if and only if  $\mathfrak{M}' \models_L \alpha'$ , where  $\mathfrak{M}'$  is the  $\mathcal{L}'$ -**structure** corresponding to  $\mathcal{L}$  and  $\alpha' \in L(\mathcal{L}')$ .
5. (*Boolean Property*) The abstract logic  $\langle L, \models_L \rangle$  is closed under the Boolean connectives in the sense that for each  $\alpha \in L(\mathcal{L})$  there is a  $\beta \in L(\mathcal{L})$  such that  $\mathfrak{M} \models_L \beta$  if and only if  $\mathfrak{M} \not\models_L \alpha$ , and for each  $\alpha$  and  $\beta$  there is a  $\gamma$  such that  $\text{Mod}_L(\gamma) = \text{Mod}_L(\alpha) \cap \text{Mod}_L(\beta)$ . Similarly for atomic **formulas** and the other connectives.

6. (*Quantifier Property*) For each constant  $c$  in  $\mathcal{L}$  and  $\alpha \in L(\mathcal{L})$  there is a  $\beta \in L(\mathcal{L})$  such that

$$\text{Mod}_L^{\mathcal{L}'}(\beta) = \{\mathfrak{M} : (\mathfrak{M}, a)\} \in \text{Mod}_L^{\mathcal{L}}(\alpha) \text{ for some } a \in |\mathfrak{M}|,$$

where  $\mathcal{L}' = \mathcal{L} \setminus \{c\}$  and  $(\mathfrak{M}, a)$  is the expansion of  $\mathfrak{M}$  to  $\mathcal{L}$  assigning  $a$  to  $c$ .

7. (*Relativization Property*) Given a **sentence**  $\alpha \in L(\mathcal{L})$  and symbols  $R, c_1, \dots, c_n$  not in  $\mathcal{L}$ , there is a **sentence**  $\beta \in L(\mathcal{L} \cup \{R, c_1, \dots, c_n\})$  called the *relativization* of  $\alpha$  to  $R(x, c_1, \dots, c_n)$ , such that for each **structure**  $\mathfrak{M}$ :

$$(\mathfrak{M}, X, b_1, \dots, b_n) \models_L \beta \text{ if and only if } \mathfrak{N} \models_L \alpha,$$

where  $\mathfrak{N}$  is the substructure of  $\mathfrak{M}$  with **domain**  $|\mathfrak{N}| = \{a \in |\mathfrak{M}| : R^{\mathfrak{M}}(a, b_1, \dots, b_n)\}$  (see ??), and  $(\mathfrak{M}, X, b_1, \dots, b_n)$  is the expansion of  $\mathfrak{M}$  interpreting  $R, c_1, \dots, c_n$  by  $X, b_1, \dots, b_n$ , respectively (with  $X \subseteq M^{n+1}$ ).

**Definition lin.4.** Given two abstract logics  $\langle L_1, \models_{L_1} \rangle$  and  $\langle L_2, \models_{L_2} \rangle$  we say that the latter is *at least as expressive* as the former, written  $\langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle$ , if for each **language**  $\mathcal{L}$  and **sentence**  $\alpha \in L_1(\mathcal{L})$  there is a **sentence**  $\beta \in L_2(\mathcal{L})$  such that  $\text{Mod}_{L_1}^{\mathcal{L}}(\alpha) = \text{Mod}_{L_2}^{\mathcal{L}}(\beta)$ . The logics  $\langle L_1, \models_{L_1} \rangle$  and  $\langle L_2, \models_{L_2} \rangle$  are *equivalent* if  $\langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle$  and  $\langle L_2, \models_{L_2} \rangle \leq \langle L_1, \models_{L_1} \rangle$ .

*Remark 1.* First-order logic, i.e., the abstract logic  $\langle F, \models \rangle$ , is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a **sentence**  $\alpha$  to  $R(x, c_1, \dots, c_n)$  is obtained by replacing each **subformula**  $\forall x \beta$  by  $\forall x (R(x, c_1, \dots, c_n) \rightarrow \beta)$ . Moreover, if  $\langle L, \models_L \rangle$  is normal, then  $\langle F, \models \rangle \leq \langle L, \models_L \rangle$ , as can be shown by induction on first-order **formulas**. Accordingly, with no loss in generality, we can assume that every first-order **sentence** belongs to every normal logic.

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## Bibliography