1 Separation of Sentences

A bit of groundwork is needed before we can proceed with the proof of the interpolation theorem. An interpolant for \( \phi \) and \( \psi \) is a sentence \( \chi \) such that \( \phi \models \chi \) and \( \chi \models \psi \). By contraposition, the latter is true iff \( \neg \psi \models \neg \chi \). A sentence \( \chi \) with this property is said to separate \( \phi \) and \( \neg \psi \). So finding an interpolant for \( \phi \) and \( \psi \) amounts to finding a sentence that separates \( \phi \) and \( \neg \psi \). As so often, it will be useful to consider a generalization: a sentence that separates two sets of sentences.

**Definition int.1.** A sentence \( \chi \) separates sets of sentences \( \Gamma \) and \( \Delta \) if and only if \( \Gamma \models \chi \) and \( \Delta \models \neg \chi \). If no such sentence exists, then \( \Gamma \) and \( \Delta \) are inseparable.

The inclusion relations between the classes of models of \( \Gamma \), \( \Delta \) and \( \chi \) are represented below:

![Diagram](image.png)

Figure 1: \( \chi \) separates \( \Gamma \) and \( \Delta \)

**Lemma int.2.** Suppose \( \mathcal{L}_0 \) is the language containing every constant symbol, function symbol and predicate symbol (other than \( \approx \)) that occurs in both \( \Gamma \) and \( \Delta \), and let \( \mathcal{L}_0' \) be obtained by the addition of infinitely many new constant symbols \( c_n \) for \( n \geq 0 \). Then if \( \Gamma \) and \( \Delta \) are inseparable in \( \mathcal{L}_0 \), they are also inseparable in \( \mathcal{L}_0' \).

**Proof.** We proceed indirectly: suppose by way of contradiction that \( \Gamma \) and \( \Delta \) are separated in \( \mathcal{L}_0' \). Then \( \Gamma \models \chi[c/x] \) and \( \Delta \models \neg \chi[c/x] \) for some \( \chi \in \mathcal{L}_0 \) (where \( c \) is a new constant symbol—the case where \( \chi \) contains more than one such new constant symbol is similar). By compactness, there are finite subsets \( \Gamma_0 \) of \( \Gamma \) and \( \Delta_0 \) of \( \Delta \) such that \( \Gamma_0 \models \chi[c/x] \) and \( \Delta_0 \models \neg \chi[c/x] \). Let \( \gamma \) be the conjunction of all formulas in \( \Gamma_0 \) and \( \delta \) the conjunction of all formulas in \( \Delta_0 \). Then

\[
\gamma \models \chi[c/x], \quad \delta \models \neg \chi[c/x].
\]

From the former, by Generalization, we have \( \gamma \models \forall x \chi \), and from the latter by contraposition, \( \chi[c/x] \models \neg \delta \), whence also \( \forall x \chi \models \neg \delta \). Contraposition again gives \( \delta \models \neg \forall x \chi \). By monotony,

\[
\Gamma \models \forall x \chi, \quad \Delta \models \neg \forall x \chi,
\]

so that \( \forall x \chi \) separates \( \Gamma \) and \( \Delta \) in \( \mathcal{L}_0 \). \qed

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Lemma int.3. Suppose that $\Gamma \cup \{\exists x \sigma\}$ and $\Delta$ are inseparable, and $c$ is a new constant symbol not in $\Gamma$, $\Delta$, or $\sigma$. Then $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and $\Delta$ are also inseparable.

Proof. Suppose for contradiction that $\chi$ separates $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and $\Delta$, while at the same time $\Gamma \cup \{\exists x \sigma\}$ and $\Delta$ are inseparable. We distinguish two cases:

1. $c$ does not occur in $\chi$; in this case $\Gamma \cup \{\exists x \sigma, \neg \chi\}$ is satisfiable (otherwise $\chi$ separates $\Gamma \cup \{\exists x \sigma\}$ and $\Delta$). It remains so if $\sigma[c/x]$ is added, so $\chi$ does not separate $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and $\Delta$ after all.

2. $c$ does occur in $\chi$ so that $\chi$ has the form $\chi[c/x]$. Then we have that $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\} \models \chi[c/x],$$

whence $\exists x \sigma \vdash \forall x (\sigma \rightarrow \chi)$ by the Deduction Theorem and Generalization, and finally $\Gamma \cup \{\exists x \sigma\} \models \exists x \chi$. On the other hand, $\Delta \vdash \neg \chi[c/x]$ and hence by Generalization $\Delta \vdash \neg \exists x \chi$. So $\Gamma \cup \{\exists x \sigma\}$ and $\Delta$ are separable, a contradiction. □

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Bibliography