

int.1 Separation of Sentences

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sec

A bit of groundwork is needed before we can proceed with the proof of the interpolation theorem. An interpolant for φ and ψ is a sentence χ such that $\varphi \models \chi$ and $\chi \models \psi$. By contraposition, the latter is true iff $\neg\psi \models \neg\chi$. A sentence χ with this property is said to *separate* φ and $\neg\psi$. So finding an interpolant for φ and ψ amounts to finding a sentence that separates φ and $\neg\psi$. As so often, it will be useful to consider a generalization: a sentence that separates two sets of sentences.

Definition int.1. A sentence χ *separates* sets of sentences Γ and Δ if and only if $\Gamma \models \chi$ and $\Delta \models \neg\chi$. If no such sentence exists, then Γ and Δ are *inseparable*.

The inclusion relations between the classes of models of Γ , Δ and χ are represented below:

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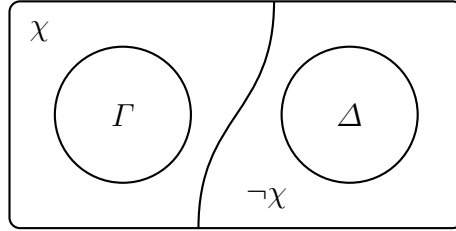


Figure 1: χ separates Γ and Δ

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lem:sep1

Lemma int.2. Suppose \mathcal{L}_0 is the language containing every constant symbol, function symbol and predicate symbol (other than \doteq) that occurs in both Γ and Δ , and let \mathcal{L}'_0 be obtained by the addition of infinitely many new constant symbols c_n for $n \geq 0$. Then if Γ and Δ are inseparable in \mathcal{L}_0 , they are also inseparable in \mathcal{L}'_0 .

Proof. We proceed indirectly: suppose by way of contradiction that Γ and Δ are separated in \mathcal{L}'_0 . Then $\Gamma \models \chi[c/x]$ and $\Delta \models \neg\chi[c/x]$ for some $\chi \in \mathcal{L}_0$ (where c is a new constant symbol—the case where χ contains more than one such new constant symbol is similar). By compactness, there are finite subsets Γ_0 of Γ and Δ_0 of Δ such that $\Gamma_0 \models \chi[c/x]$ and $\Delta_0 \models \neg\chi[c/x]$. Let γ be the conjunction of all formulas in Γ_0 and δ the conjunction of all formulas in Δ_0 . Then

$$\gamma \models \chi[c/x], \quad \delta \models \neg\chi[c/x].$$

From the former, by Generalization, we have $\gamma \models \forall x \chi$, and from the latter by contraposition, $\chi[c/x] \models \neg\delta$, whence also $\forall x \chi \models \neg\delta$. Contraposition again gives $\delta \models \neg\forall x \chi$. By monotony,

$$\Gamma \models \forall x \chi, \quad \Delta \models \neg\forall x \chi,$$

so that $\forall x \chi$ separates Γ and Δ in \mathcal{L}_0 . □

Lemma int.3. Suppose that $\Gamma \cup \{\exists x \sigma\}$ and Δ are inseparable, and c is a new constant symbol not in Γ , Δ , or σ . Then $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and Δ are also inseparable. mod:int:sep:
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Proof. Suppose for contradiction that χ separates $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and Δ , while at the same time $\Gamma \cup \{\exists x \sigma\}$ and Δ are inseparable. We distinguish two cases:

1. c does not occur in χ : in this case $\Gamma \cup \{\exists x \sigma, \neg \chi\}$ is satisfiable (otherwise χ separates $\Gamma \cup \{\exists x \sigma\}$ and Δ). It remains so if $\sigma[c/x]$ is added, so χ does not separate $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and Δ after all.
2. c does occur in χ so that χ has the form $\chi[c/x]$. Then we have that

$$\Gamma \cup \{\exists x \sigma, \sigma[c/x]\} \models \chi[c/x],$$

whence $\Gamma, \exists x \sigma \models \forall x (\sigma \rightarrow \chi)$ by the Deduction Theorem and Generalization, and finally $\Gamma \cup \{\exists x \sigma\} \models \exists x \chi$. On the other hand, $\Delta \models \neg \chi[c/x]$ and hence by Generalization $\Delta \models \neg \exists x \chi$. So $\Gamma \cup \{\exists x \sigma\}$ and Δ are separable, a contradiction. □

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Bibliography