

## int.1 Separation of Sentences

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sec

A bit of groundwork is needed before we can proceed with the proof of the interpolation theorem. An interpolant for  $\varphi$  and  $\psi$  is a sentence  $\chi$  such that  $\varphi \models \chi$  and  $\chi \models \psi$ . By contraposition, the latter is true iff  $\neg\psi \models \neg\chi$ . A sentence  $\chi$  with this property is said to *separate*  $\varphi$  and  $\neg\psi$ . So finding an interpolant for  $\varphi$  and  $\psi$  amounts to finding a sentence that separates  $\varphi$  and  $\neg\psi$ . As so often, it will be useful to consider a generalization: a sentence that separates two sets of sentences.

**Definition int.1.** A sentence  $\chi$  *separates* sets of sentences  $\Gamma$  and  $\Delta$  if and only if  $\Gamma \models \chi$  and  $\Delta \models \neg\chi$ . If no such sentence exists, then  $\Gamma$  and  $\Delta$  are *inseparable*.

The inclusion relations between the classes of models of  $\Gamma$ ,  $\Delta$  and  $\chi$  are represented below:

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fig:sep

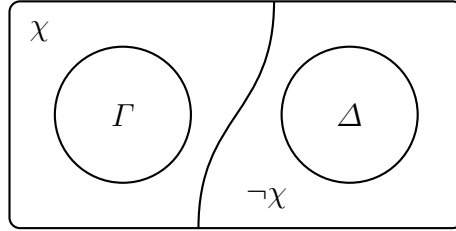


Figure 1:  $\chi$  separates  $\Gamma$  and  $\Delta$

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lem:sep1

**Lemma int.2.** Suppose  $\mathcal{L}_0$  is the language containing every constant symbol, function symbol and predicate symbol (other than  $\doteq$ ) that occurs in both  $\Gamma$  and  $\Delta$ , and let  $\mathcal{L}'_0$  be obtained by the addition of infinitely many new constant symbols  $c_n$  for  $n \geq 0$ . Then if  $\Gamma$  and  $\Delta$  are inseparable in  $\mathcal{L}_0$ , they are also inseparable in  $\mathcal{L}'_0$ .

*Proof.* We proceed indirectly: suppose by way of contradiction that  $\Gamma$  and  $\Delta$  are separated in  $\mathcal{L}'_0$ . Then  $\Gamma \models \chi[c/x]$  and  $\Delta \models \neg\chi[c/x]$  for some  $\chi \in \mathcal{L}_0$  (where  $c$  is a new constant symbol—the case where  $\chi$  contains more than one such new constant symbol is similar). By compactness, there are finite subsets  $\Gamma_0$  of  $\Gamma$  and  $\Delta_0$  of  $\Delta$  such that  $\Gamma_0 \models \chi[c/x]$  and  $\Delta_0 \models \neg\chi[c/x]$ . Let  $\gamma$  be the conjunction of all formulas in  $\Gamma_0$  and  $\delta$  the conjunction of all formulas in  $\Delta_0$ . Then

$$\gamma \models \chi[c/x], \quad \delta \models \neg\chi[c/x].$$

From the former, by Generalization, we have  $\gamma \models \forall x \chi$ , and from the latter by contraposition,  $\chi[c/x] \models \neg\delta$ , whence also  $\forall x \chi \models \neg\delta$ . Contraposition again gives  $\delta \models \neg\forall x \chi$ . By monotony,

$$\Gamma \models \forall x \chi, \quad \Delta \models \neg\forall x \chi,$$

so that  $\forall x \chi$  separates  $\Gamma$  and  $\Delta$  in  $\mathcal{L}_0$ . □

**Lemma int.3.** Suppose that  $\Gamma \cup \{\exists x \sigma\}$  and  $\Delta$  are inseparable, and  $c$  is a new constant symbol not in  $\Gamma$ ,  $\Delta$ , or  $\sigma$ . Then  $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$  and  $\Delta$  are also inseparable. mod:int:sep:  
lem:sep2

*Proof.* Suppose for contradiction that  $\chi$  separates  $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$  and  $\Delta$ , while at the same time  $\Gamma \cup \{\exists x \sigma\}$  and  $\Delta$  are inseparable. We distinguish two cases:

1.  $c$  does not occur in  $\chi$ : in this case  $\Gamma \cup \{\exists x \sigma, \neg \chi\}$  is satisfiable (otherwise  $\chi$  separates  $\Gamma \cup \{\exists x \sigma\}$  and  $\Delta$ ). It remains so if  $\sigma[c/x]$  is added, so  $\chi$  does not separate  $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$  and  $\Delta$  after all.
2.  $c$  does occur in  $\chi$  so that  $\chi$  has the form  $\chi[c/x]$ . Then we have that

$$\Gamma \cup \{\exists x \sigma, \sigma[c/x]\} \models \chi[c/x],$$

whence  $\Gamma, \exists x \sigma \models \forall x (\sigma \rightarrow \chi)$  by the Deduction Theorem and Generalization, and finally  $\Gamma \cup \{\exists x \sigma\} \models \exists x \chi$ . On the other hand,  $\Delta \models \neg \chi[c/x]$  and hence by Generalization  $\Delta \models \neg \exists x \chi$ . So  $\Gamma \cup \{\exists x \sigma\}$  and  $\Delta$  are separable, a contradiction. □

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## Bibliography