Chapter udf

The Interpolation Theorem

int.1 Introduction

The interpolation theorem is the following result: Suppose $\models \varphi \rightarrow \psi$. Then there is a sentence $\chi$ such that $\models \varphi \rightarrow \chi$ and $\models \chi \rightarrow \psi$. Moreover, every constant symbol, function symbol, and predicate symbol (other than $=$) in $\chi$ occurs both in $\varphi$ and $\psi$. The sentence $\chi$ is called an interpolant of $\varphi$ and $\psi$.

The interpolation theorem is interesting in its own right, but its main importance lies in the fact that it can be used to prove results about definability in a theory, and the conditions under which combining two consistent theories results in a consistent theory. The first result is known as the Beth definability theorem; the second, Robinson’s joint consistency theorem.

int.2 Separation of Sentences

A bit of groundwork is needed before we can proceed with the proof of the interpolation theorem. An interpolant for $\varphi$ and $\psi$ is a sentence $\chi$ such that $\varphi \models \chi$ and $\chi \models \psi$. By contraposition, the latter is true iff $\neg \psi \models \neg \chi$. A sentence $\chi$ with this property is said to separate $\varphi$ and $\neg \psi$. So finding an interpolant for $\varphi$ and $\psi$ amounts to finding a sentence that separates $\varphi$ and $\neg \psi$. As so often, it will be useful to consider a generalization: a sentence that separates two sets of sentences.

Definition int.1. A sentence $\chi$ separates sets of sentences $\Gamma$ and $\Delta$ if and only if $\Gamma \models \chi$ and $\Delta \models \neg \chi$. If no such sentence exists, then $\Gamma$ and $\Delta$ are inseparable.

The inclusion relations between the classes of models of $\Gamma$, $\Delta$ and $\chi$ are represented below:

Lemma int.2. Suppose $\mathcal{L}_0$ is the language containing every constant symbol, function symbol and predicate symbol (other than $=$) that occurs in both $\Gamma$ and $\Delta$, and let $\mathcal{L}_0'$ be obtained by the addition of infinitely many new constant
symbols $c_n$ for $n \geq 0$. Then if $\Gamma$ and $\Delta$ are inseparable in $L_0$, they are also inseparable in $L'_0$.

**Proof.** We proceed indirectly: suppose by way of contradiction that $\Gamma$ and $\Delta$ are separated in $L'_0$. Then $\Gamma \models \chi[c/x]$ and $\Delta \models \neg \chi[c/x]$ for some $\chi \in L_0$ (where $c$ is a new constant symbol—the case where $\chi$ contains more than one such new constant symbol is similar). By compactness, there are finite subsets $\Gamma_0$ of $\Gamma$ and $\Delta_0$ of $\Delta$ such that $\Gamma_0 \models \chi[c/x]$ and $\Delta_0 \models \neg \chi[c/x]$. Let $\gamma$ be the conjunction of all formulas in $\Gamma_0$ and $\delta$ the conjunction of all formulas in $\Delta_0$. Then

$$
\gamma \models \chi[c/x], \quad \delta \models \neg \chi[c/x].
$$

From the former, by Generalization, we have $\gamma \models \forall x \chi$, and from the latter by contraposition, $\chi[c/x] \models \neg \delta$, whence also $\forall x \chi \models \neg \delta$. Contraposition again gives $\delta \models \neg \forall x \chi$. By monotony,

$$
\Gamma \models \forall x \chi, \quad \Delta \models \neg \forall x \chi,
$$

so that $\forall x \chi$ separates $\Gamma$ and $\Delta$ in $L_0$.

**Lemma int.3.** Suppose that $\Gamma \cup \{\exists x \sigma\}$ and $\Delta$ are inseparable, and $c$ is a new constant symbol not in $\Gamma$, $\Delta$, or $\sigma$. Then $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and $\Delta$ are also inseparable.

**Proof.** Suppose for contradiction that $\chi$ separates $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and $\Delta$, while at the same time $\Gamma \cup \{\exists x \sigma\}$ and $\Delta$ are inseparable. We distinguish two cases:

1. $c$ does not occur in $\chi$: in this case $\Gamma \cup \{\exists x \sigma, \neg \chi\}$ is satisfiable (otherwise $\chi$ separates $\Gamma \cup \{\exists x \sigma\}$ and $\Delta$). It remains so if $\sigma[c/x]$ is added, so $\chi$ does not separate $\Gamma \cup \{\exists x \sigma, \sigma[c/x]\}$ and $\Delta$ after all.

2. $c$ does occur in $\chi$ so that $\chi$ has the form $\chi[c/x]$. Then we have that

$$
\Gamma \cup \{\exists x \sigma, \sigma[c/x]\} \models \chi[c/x],
$$

whence $\Gamma, \exists x \sigma \models \forall x (\sigma \rightarrow \chi)$ by the Deduction Theorem and Generalization, and finally $\Gamma \cup \{\exists x \sigma\} \models \exists x \chi$. On the other hand, $\Delta \models \neg \chi[c/x]$ and hence by Generalization $\Delta \models \neg \exists x \chi$. So $\Gamma \cup \{\exists x \sigma\}$ and $\Delta$ are separable, a contradiction. \[\square\]
Craig’s Interpolation Theorem

Theorem int.4 (Craig’s Interpolation Theorem). If $\vdash \varphi \rightarrow \psi$, then there is a sentence $\chi$ such that $\vdash \varphi \rightarrow \chi$ and $\vdash \chi \rightarrow \psi$, and every constant symbol, function symbol, and predicate symbol (other than $=$) in $\chi$ occurs both in $\varphi$ and $\psi$. The sentence $\chi$ is called an interpolant of $\varphi$ and $\psi$.

Proof. Suppose $L_1$ is the language of $\varphi$ and $L_2$ is the language of $\psi$. Let $L_0 = L_1 \cap L_2$. For each $i \in \{0, 1, 2\}$, let $L'_i$ be obtained from $L_i$ by adding the infinitely many new constant symbols $c_0, c_1, c_2, \ldots$.

If $\varphi$ is unsatisfiable, $\exists x \ x \neq x$ is an interpolant. If $\neg \psi$ is unsatisfiable (and hence $\psi$ is valid), $\exists x \ x = x$ is an interpolant. So we may assume also that both $\varphi$ and $\neg \psi$ are satisfiable.

In order to prove the contrapositive of the Interpolation Theorem, assume that there is no interpolant for $\varphi$ and $\psi$. In other words, assume that $\{\varphi\}$ and $\{\neg \psi\}$ are inseparable in $L_0$.

Our goal is to extend the pair $\{\varphi\}, \{\neg \psi\}$ to a maximally inseparable pair $(\Gamma^*, \Delta^*)$. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ enumerate the sentences of $L_1$, and $\psi_0, \psi_1, \psi_2, \ldots$ enumerate the sentences of $L_2$. We define two increasing sequences of sets of sentences $(\Gamma_n, \Delta_n)$, for $n \geq 0$, as follows. Put $\Gamma_0 = \{\varphi\}$ and $\Delta_0 = \{\neg \psi\}$.

Assuming $(\Gamma'_n, \Delta_n)$ are already defined, define $\Gamma_{n+1}$ and $\Delta_{n+1}$ by:

1. If $\Gamma_n \cup \{\varphi_n\}$ and $\Delta_n$ are inseparable in $L'_0$, put $\varphi_n$ in $\Gamma_{n+1}$. Moreover, if $\varphi_n$ is an existential formula $\exists x \sigma$ then pick a new constant symbol $c$ not occurring in $\Gamma_n$, $\Delta_n$, $\varphi_n$ or $\psi_n$, and put $\sigma[c/x]$ in $\Gamma_{n+1}$.

2. If $\Gamma_{n+1}$ and $\Delta_n \cup \{\psi_n\}$ are inseparable in $L'_0$, put $\psi_n$ in $\Delta_{n+1}$. Moreover, if $\psi_n$ is an existential formula $\exists x \sigma$, then pick a new constant symbol $c$ not occurring in $\Gamma_{n+1}$, $\Delta_n$, $\varphi_n$ or $\psi_n$, and put $\sigma[c/x]$ in $\Delta_{n+1}$.

Finally, define:

$$
\Gamma^* = \bigcup_{n \geq 0} \Gamma_n, \quad \Delta^* = \bigcup_{n \geq 0} \Delta_n.
$$

By simultaneous induction on $n$ we can now prove:

1. $\Gamma_n$ and $\Delta_n$ are inseparable in $L'_0$;

2. $\Gamma_{n+1}$ and $\Delta_n$ are inseparable in $L'_0$.

The basis for (1) is given by Lemma int.2. For part (2), we need to distinguish three cases:

1. If $\Gamma_0 \cup \{\varphi_0\}$ and $\Delta_0$ are separable, then $\Gamma_1 = \Gamma_0$ and (2) is just (1);

2. If $\Gamma_0 = \Gamma_0 \cup \{\varphi_0\}$, then $\Gamma_1$ and $\Delta_0$ are inseparable by construction.
3. It remains to consider the case where $\varphi_0$ is existential, so that $\Gamma_1 = \Gamma_0 \cup \{\exists x \sigma[c/x]\}$. By construction, $\Gamma_0 \cup \{\exists x \sigma\}$ and $\Delta_0$ are inseparable, so that by Lemma int.3 also $\Gamma_0 \cup \{\exists x \sigma[c/x]\}$ and $\Delta_0$ are inseparable.

This completes the basis of the induction for (1) and (2) above. Now for the inductive step. For (1), if $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ then $\Gamma_{n+1}$ and $\Delta_{n+1}$ are inseparable by construction (even when $\psi_n$ is existential, by Lemma int.3); if $\Delta_{n+1} = \Delta_n$ (because $\Gamma_{n+1}$ and $\Delta_n \cup \{\psi_n\}$ are separable), then we use the induction hypothesis on (2). For the inductive step for (2), if $\Gamma_{n+2} = \Gamma_{n+1} \cup \{\varphi_{n+1}\}$ then $\Gamma_{n+2}$ and $\Delta_{n+1}$ are inseparable by construction (even when $\varphi_{n+1}$ is existential, by Lemma int.3); and if $\Gamma_{n+2} = \Gamma_{n+1}$ then we use the inductive case for (1) just proved. This concludes the induction on (1) and (2).

It follows that $\Gamma^*$ and $\Delta^*$ are inseparable; if not, by compactness, there is $n \geq 0$ that separates $\Gamma_n$ and $\Delta_n$, against (1). In particular, $\Gamma^*$ and $\Delta^*$ are consistent: for if the former or the latter is inconsistent, then they are separated by $\exists x x \neq x$ or $\forall x x = x$, respectively.

We now show that $\Gamma^*$ is maximally consistent in $L'_1$ and likewise $\Delta^*$ in $L'_2$. For the former, suppose that $\varphi_n \notin \Gamma^*$ and $\neg \varphi_n \notin \Gamma^*$, for some $n \geq 0$. If $\varphi_n \notin \Gamma^*$ then $\Gamma_0 \cup \{\varphi_n\}$ is separable from $\Delta_n$, and so there is $\chi \in L'_0$ such that both:

$$\Gamma^* \models \varphi_n \rightarrow \chi, \quad \Delta^* \models \neg \chi.$$

Likewise, if $\neg \varphi_n \notin \Gamma^*$, there is $\chi' \in L'_0$ such that both:

$$\Gamma^* \models \neg \varphi_n \rightarrow \chi', \quad \Delta^* \models \neg \chi'.$$

By propositional logic, $\Gamma^* \models \chi \lor \chi'$ and $\Delta^* \models \neg (\chi \lor \chi')$, so $\chi \lor \chi'$ separates $\Gamma^*$ and $\Delta^*$. A similar argument establishes that $\Delta^*$ is maximal.

Finally, we show that $\Gamma^* \cap \Delta^*$ is maximally consistent in $L'_0$. It is obviously consistent, since it is the intersection of consistent sets. To show maximality, let $\sigma \in L'_0$. Now, $\Gamma^*$ is maximal in $L'_1 \supseteq L'_0$, and similarly $\Delta^*$ is maximal in $L'_2 \supseteq L'_0$. It follows that either $\sigma \in \Gamma^*$ or $\neg \sigma \in \Gamma^*$, and either $\sigma \in \Delta^*$ or $\neg \sigma \in \Delta^*$. If $\sigma \in \Gamma^*$ and $\neg \sigma \in \Delta^*$, then $\sigma$ would separate $\Gamma^*$ and $\Delta^*$; and if $\neg \sigma \in \Gamma^*$ and $\sigma \in \Delta^*$ then $\Gamma^*$ and $\Delta^*$ would be separated by $\neg \sigma$. Hence, either $\sigma \in \Gamma^* \cap \Delta^*$ or $\neg \sigma \in \Gamma^* \cap \Delta^*$, and $\Gamma^* \cap \Delta^*$ is maximal.

Since $\Gamma^*$ is maximally consistent, it has a model $\mathfrak{M}_1'$ whose domain $|\mathfrak{M}_1'|$ comprises all and only the elements $c^\mathfrak{M}_1'$ interpreting the constant symbols—just like in the proof of the completeness theorem (??). Similarly, $\Delta^*$ has a model $\mathfrak{M}_2'$ whose domain $|\mathfrak{M}_2'|$ is given by the interpretations $c^\mathfrak{M}_2'$ of the constant symbols.

Let $\mathfrak{M}_1$ be obtained from $\mathfrak{M}_1'$ by dropping interpretations for constant symbols, function symbols, and predicate symbols in $L'_1 \setminus L'_0$, and similarly for $\mathfrak{M}_2$. Then the map $h: M_1 \rightarrow M_2$ defined by $h(c^\mathfrak{M}_1') = c^\mathfrak{M}_2'$ is an isomorphism in $L'_0$, because $\Gamma^* \cap \Delta^*$ is maximally consistent in $L'_0$, as shown. This follows because any $L'_0$-sentence either belongs to both $\Gamma^*$ and $\Delta^*$, or to neither: so
Interpolation Theorem.

whence

\text{interpolation}

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only if there is a formula \( \chi \)

\text{Definition int.5.}

Suppose \( P \)

\text{Definition int.6.}

Suppose \( P \)

\text{simply.}

\text{entails) a formalized explicit definition, i.e.,}

\text{similarly.}

\text{theory. To define an}

\text{definition of}

\text{in this way—whenever it}

\text{a relation. A theory may also be such that the interpretation}

\text{entails} \( \chi \)

\text{of \( P \) in terms of} \( \chi \).

\text{We can then say also that a theory} \( \Sigma \)

\text{in a language} containing

\text{predicate symbol} \( P \) explicitly defines \( P \) if it contains (or at least entails) a formalized explicit definition, i.e.,

\text{formulas} that \( \mathfrak{M} \) agrees with \( \mathfrak{M}_1 \) on all

\text{formulas} of \( \mathcal{L}_1 \) and with \( \mathfrak{M}_2 \) on all

\text{formulas} of \( \mathcal{L}_2 \). In particular, \( \mathfrak{M} \models \Gamma^* \cup \Delta^* \),

whence \( \mathfrak{M} \models \varphi \) and \( \mathfrak{M} \models \neg \psi \), and \( \not\models \varphi \rightarrow \psi \). This concludes the proof of Craig’s

Interpolation Theorem.

\text{int.4 The Definability Theorem}

One important application of the interpolation theorem is Beth’s definability theorem. To define an \( n \)-place relation \( R \) we can give a \text{formula} \( \chi \) with \( n \) free variables which does not involve \( R \). This would be an \text{explicit} definition of \( R \) in terms of \( \chi \). We can then say also that a theory \( \Sigma(P) \) in a language containing the \( n \)-place predicate symbol \( P \) explicitly defines \( P \) if it contains (or at least entails) a formalized explicit definition, i.e.,

\[ \Sigma(P) \models \forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)). \]

But an explicit definition is only one way of defining—in the sense of determining completely—a relation. A theory may also be such that the interpretation of \( P \) is fixed by the interpretation of the rest of the language in any model. The definability theorem states that whenever a theory fixes the interpretation of \( P \) in this way—whenever it \text{implicitly defines} \( P \)—then it also explicitly defines it.

\text{Definition int.5.} Suppose \( \mathcal{L} \) is a language not containing the \text{predicate symbol} \( P \). A set \( \Sigma(P) \) of \text{sentences} of \( \mathcal{L} \cup \{ P \} \) \text{explicitly defines} \( P \) if and only if there is a \text{formula} \( \chi(x_1, \ldots, x_n) \) of \( \mathcal{L} \) such that

\[ \Sigma(P) \models \forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)). \]

\text{Definition int.6.} Suppose \( \mathcal{L} \) is a language not containing the \text{predicate symbols} \( P \) and \( P' \). A set \( \Sigma(P) \) of \text{sentences} of \( \mathcal{L} \cup \{ P \} \) \text{implicitly defines} \( P \) if and only if

\[ \Sigma(P) \cup \Sigma(P') \models \forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_n) \leftrightarrow P'(x_1, \ldots, x_n)). \]
where $\Sigma(P')$ is the result of uniformly replacing $P$ with $P'$ in $\Sigma(P)$.

In other words, for any model $\mathfrak{M}$ and $R, R' \subseteq |\mathfrak{M}|^n$, if both $(\mathfrak{M}, R) \models \Sigma(P)$ and $(\mathfrak{M}, R') \models \Sigma(P')$, then $R = R'$; where $(\mathfrak{M}, R)$ is the structure $\mathfrak{M}'$ for the expansion of $\mathcal{L}$ to $\mathcal{L} \cup \{P\}$ such that $P^{\mathfrak{M}'} = R$, and similarly for $(\mathfrak{M}, R')$.

**Theorem int.7 (Beth Definability Theorem).** A set $\Sigma(P)$ of $\mathcal{L} \cup \{P\}$-formulas implicitly defines $P$ if and only $\Sigma(P)$ explicitly defines $P$.

**Proof.** If $\Sigma(P)$ explicitly defines $P$ then both

$$\Sigma(P) \models \forall x_1 \ldots \forall x_n \left(P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)\right)$$

$$\Sigma(P') \models \forall x_1 \ldots \forall x_n \left(P'(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)\right)$$

and the conclusion follows. For the converse: assume that $\Sigma(P)$ implicitly defines $P$. First, we add constant symbols $c_1, \ldots, c_n$ to $\mathcal{L}$. Then

$$\Sigma(P) \cup \Sigma(P') \models P(c_1, \ldots, c_n) \rightarrow P'(c_1, \ldots, c_n).$$

By compactness, there are finite sets $\Delta_0 \subseteq \Sigma(P)$ and $\Delta_1 \subseteq \Sigma(P')$ such that

$$\Delta_0 \cup \Delta_1 \models P(c_1, \ldots, c_n) \rightarrow P'(c_1, \ldots, c_n).$$

Let $\theta(P)$ be the conjunction of all sentences $\varphi(P)$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$ and let $\theta(P')$ be the conjunction of all sentences $\varphi(P')$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$. Then $\theta(P) \land \theta(P') \models P(c_1, \ldots, c_n) \rightarrow P'c_1 \ldots c_n$. We can re-arrange this so that each predicate symbol occurs on one side of $\models$:

$$\theta(P) \land P(c_1, \ldots, c_n) \models \theta(P') \models P'(c_1, \ldots, c_n).$$

By Craig’s Interpolation Theorem there is a sentence $\chi(c_1, \ldots, c_n)$ not containing $P$ or $P'$ such that:

$$\theta(P) \land P(c_1, \ldots, c_n) \models \chi(c_1, \ldots, c_n); \quad \chi(c_1, \ldots, c_n) \models \theta(P') \models P'(c_1, \ldots, c_n).$$

From the former of these two entailments we have: $\theta(P) \models P(c_1, \ldots, c_n) \rightarrow \chi(c_1, \ldots, c_n)$, and from the latter, since an $\mathcal{L} \cup \{P\}$-model $(\mathfrak{M}, R) \models \varphi(P)$ if and only if the corresponding $\mathcal{L} \cup \{P\}$-model $(\mathfrak{M}, R) \models \varphi(P')$, we have $\chi(c_1, \ldots, c_n) \models \theta(P) \models P'(c_1, \ldots, c_n)$, from which:

$$\theta(P) \models \chi(c_1, \ldots, c_n) \rightarrow P(c_1, \ldots, c_n).$$

Putting the two together, $\theta(P) \models P(c_1, \ldots, c_n) \leftrightarrow \chi(c_1, \ldots, c_n)$, and by monotony and generalization also

$$\Sigma(P) \models \forall x_1 \ldots \forall x_n \left(P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)\right). \quad \square$$
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