Chapter udf

The Interpolation Theorem

int.1 Introduction

mod:int:int:sec The interpolation theorem is the following result: Suppose $\vDash \varphi \rightarrow \psi$. Then there is a sentence χ such that $\vDash \varphi \rightarrow \chi$ and $\vDash \chi \rightarrow \psi$. Moreover, every constant symbol, function symbol, and predicate symbol (other than $=$) in χ occurs both in φ and ψ. The sentence χ is called an *interpolant* of φ and ψ.

The interpolation theorem is interesting in its own right, but its main importance lies in the fact that it can be used to prove results about definability in a theory, and the conditions under which combining two consistent theories results in a consistent theory. The first result is known as the Beth definability theorem; the second, Robinson's joint consistency theorem.

int.2 Separation of Sentences

mod:int:sep:sec A bit of groundwork is needed before we can proceed with the proof of the interpolation theorem. An interpolant for φ and ψ is a sentence χ such that $\varphi \vDash \chi$ and $\chi \vDash \psi$. By contraposition, the latter is true iff $\neg\psi \vDash \neg\chi$. A sentence χ with this property is said to *separate* φ and $\neg\psi$. So finding an interpolant for φ and ψ amounts to finding a sentence that separates φ and $\neg\psi$. As so often, it will be useful to consider a generalization: a sentence that separates two *sets* of sentences.

Definition int.1. A sentence χ *separates* sets of sentences Γ and Δ if and only if $\Gamma \vDash \chi$ and $\Delta \vDash \neg\chi$. If no such sentence exists, then Γ and Δ are *inseparable*.

The inclusion relations between the classes of models of Γ, Δ and χ are represented below:

mod:int:sep:lem:sep1 **Lemma int.2.** *Suppose $\mathcal{L}_0$ is the language containing every constant symbol, function symbol and predicate symbol (other than $\doteq$) that occurs in* both *Γ and Δ, and let $\mathcal{L}_0'$ be obtained by the addition of infinitely many new constant symbols c_n for $n \geq 0$. Then if Γ and Δ are inseparable in $\mathcal{L}_0$, they are also inseparable in $\mathcal{L}_0'$.*

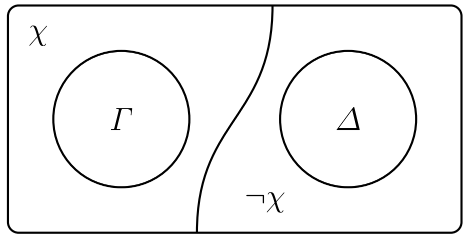

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Figure int.1: χ separates Γ and Δ

Proof. We proceed indirectly: suppose by way of contradiction that Γ and Δ are separated in $\mathcal{L}_0'$. Then $\Gamma \vDash \chi[c/x]$ and $\Delta \vDash \neg\chi[c/x]$ for some $\chi \in \mathcal{L}_0$ (where c is a new constant symbol—the case where χ contains more than one such new constant symbol is similar). By compactness, there are *finite* subsets Γ_0 of Γ and Δ_0 of Δ such that $\Gamma_0 \vDash \chi[c/x]$ and $\Delta_0 \vDash \neg\chi[c/x]$. Let γ be the conjunction of all formulas in Γ_0 and δ the conjunction of all formulas in Δ_0. Then

$$\gamma \vDash \chi[c/x], \qquad \delta \vDash \neg\chi[c/x].$$

From the former, by Generalization, we have $\gamma \vDash \forall x\, \chi$, and from the latter by contraposition, $\chi[c/x] \vDash \neg\delta$, whence also $\forall x\, \chi \vDash \neg\delta$. Contraposition again gives $\delta \vDash \neg\forall x\, \chi$. By monotony,

$$\Gamma \vDash \forall x\, \chi, \qquad \Delta \vDash \neg\forall x\, \chi,$$

so that $\forall x\, \chi$ separates Γ and Δ in $\mathcal{L}_0$. □

Lemma int.3. *Suppose that $\Gamma \cup \{\exists x\, \sigma\}$ and Δ are inseparable, and c is a new constant symbol not in Γ, Δ, or σ. Then $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ are also inseparable.* mod:int:sep:lem:sep2

Proof. Suppose for contradiction that χ separates $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ, while at the same time $\Gamma \cup \{\exists x \sigma\}$ and Δ are inseparable. We distinguish two cases:

1. c does not occur in χ: in this case $\Gamma \cup \{\exists x\, \sigma, \neg\chi\}$ is satisfiable (otherwise χ separates $\Gamma \cup \{\exists x\, \sigma\}$ and Δ). It remains so if $\sigma[c/x]$ is added, so χ does not separate $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ after all.

2. c does occur in χ so that χ has the form $\chi[c/x]$. Then we have that

$$\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\} \vDash \chi[c/x],$$

whence $\Gamma, \exists x\, \sigma \vDash \forall x\, (\sigma \rightarrow \chi)$ by the Deduction Theorem and Generalization, and finally $\Gamma \cup \{\exists x\, \sigma\} \vDash \exists x\, \chi$. On the other hand, $\Delta \vDash \neg\chi[c/x]$ and hence by Generalization $\Delta \vDash \neg\exists x\, \chi$. So $\Gamma \cup \{\exists x\, \sigma\}$ and Δ are separable, a contradiction. □

int.3 Craig's Interpolation Theorem

mod:int:prf: sec

mod:int:prf: thm:interpol

Theorem int.4 (Craig's Interpolation Theorem). *If $\vDash \varphi \to \psi$, then there is a sentence χ such that $\vDash \varphi \to \chi$ and $\vDash \chi \to \psi$, and every constant symbol, function symbol, and predicate symbol (other than $=$) in χ occurs both in φ and ψ. The sentence χ is called an* interpolant *of φ and ψ.*

Proof. Suppose $\mathcal{L}_1$ is the language of φ and $\mathcal{L}_2$ is the language of ψ. Let $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$. For each $i \in \{0, 1, 2\}$, let $\mathcal{L}'_i$ be obtained from $\mathcal{L}_i$ by adding the infinitely many new constant symbols $c_0, c_1, c_2, \dots$.

If φ is unsatisfiable, $\exists x\, x \neq x$ is an interpolant. If $\neg\psi$ is unsatisfiable (and hence ψ is valid), $\exists x\, x = x$ is an interpolant. So we may assume also that both φ and $\neg\psi$ are satisfiable.

In order to prove the contrapositive of the Interpolation Theorem, assume that there is no interpolant for φ and ψ. In other words, assume that $\{\varphi\}$ and $\{\neg\psi\}$ are inseparable in $\mathcal{L}_0$.

Our goal is to extend the pair $(\{\varphi\}, \{\neg\psi\})$ to a maximally inseparable pair (Γ^*, Δ^*). Let φ_0, φ_1, φ_2, … enumerate the sentences of $\mathcal{L}_1$, and ψ_0, ψ_1, ψ_2, … enumerate the sentences of $\mathcal{L}_2$. We define two increasing sequences of sets of sentences (Γ_n, Δ_n), for $n \geq 0$, as follows. Put $\Gamma_0 = \{\varphi\}$ and $\Delta_0 = \{\neg\psi\}$. Assuming (Γ_n, Δ_n) are already defined, define Γ_{n+1} and Δ_{n+1} by:

1. If $\Gamma_n \cup \{\varphi_n\}$ and Δ_n are inseparable in $\mathcal{L}'_0$, put φ_n in Γ_{n+1}. Moreover, if φ_n is an existential formula $\exists x\, \sigma$ then pick a new constant symbol c not occurring in Γ_n, Δ_n, φ_n or ψ_n, and put $\sigma[c/x]$ in Γ_{n+1}.

2. If Γ_{n+1} and $\Delta_n \cup \{\psi_n\}$ are inseparable in $\mathcal{L}'_0$, put ψ_n in Δ_{n+1}. Moreover, if ψ_n is an existential formula $\exists x\, \sigma$, then pick a new constant symbol c not occurring in Γ_{n+1}, Δ_n, φ_n or ψ_n, and put $\sigma[c/x]$ in Δ_{n+1}.

Finally, define:

$$\Gamma^* = \bigcup_{n \geq 0} \Gamma_n, \qquad \Delta^* = \bigcup_{n \geq 0} \Delta_n.$$

By simultaneous induction on n we can now prove:

mod:int:prf: part-a

1. Γ_n and Δ_n are inseparable in $\mathcal{L}'_0$;

mod:int:prf: part-b

2. Γ_{n+1} and Δ_n are inseparable in $\mathcal{L}'_0$.

The basis for (1) is given by Lemma int.2. For part (2), we need to distinguish three cases:

1. If $\Gamma_0 \cup \{\varphi_0\}$ and Δ_0 are separable, then $\Gamma_1 = \Gamma_0$ and (2) is just (1);

2. If $\Gamma_1 = \Gamma_0 \cup \{\varphi_0\}$, then Γ_1 and Δ_0 are inseparable by construction.

3. It remains to consider the case where φ_0 is existential, so that $\Gamma_1 = \Gamma_0 \cup \{\exists x\, \sigma, \sigma[c/x]\}$. By construction, $\Gamma_0 \cup \{\exists x\, \sigma\}$ and Δ_0 are inseparable, so that by Lemma int.3 also $\Gamma_0 \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ_0 are inseparable.

This completes the basis of the induction for (1) and (2) above. Now for the inductive step. For (1), if $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ then Γ_{n+1} and Δ_{n+1} are inseparable by construction (even when ψ_n is existential, by Lemma int.3); if $\Delta_{n+1} = \Delta_n$ (because Γ_{n+1} and $\Delta_n \cup \{\psi_n\}$ are separable), then we use the induction hypothesis on (2). For the inductive step for (2), if $\Gamma_{n+2} = \Gamma_{n+1} \cup \{\varphi_{n+1}\}$ then Γ_{n+2} and Δ_{n+1} are inseparable by construction (even when φ_{n+1} is existential, by Lemma int.3); and if $\Gamma_{n+2} = \Gamma_{n+1}$ then we use the inductive case for (1) just proved. This concludes the induction on (1) and (2).

It follows that Γ^* and Δ^* are inseparable; if not, by compactness, there is $n \geq 0$ that separates Γ_n and Δ_n, against (1). In particular, Γ^* and Δ^* are consistent: for if the former or the latter is inconsistent, then they are separated by $\exists x\, x \neq x$ or $\forall x\, x = x$, respectively.

We now show that Γ^* is maximally consistent in $\mathcal{L}_1'$ and likewise Δ^* in $\mathcal{L}_2'$. For the former, suppose that $\varphi_n \notin \Gamma^*$ and $\neg\varphi_n \notin \Gamma^*$, for some $n \geq 0$. If $\varphi_n \notin \Gamma^*$ then $\Gamma_n \cup \{\varphi_n\}$ is separable from Δ_n, and so there is $\chi \in \mathcal{L}_0'$ such that both:

$$\Gamma^* \vDash \varphi_n \rightarrow \chi, \qquad\qquad \Delta^* \vDash \neg\chi.$$

Likewise, if $\neg\varphi_n \notin \Gamma^*$, there is $\chi' \in \mathcal{L}_0'$ such that both:

$$\Gamma^* \vDash \neg\varphi_n \rightarrow \chi', \qquad\qquad \Delta^* \vDash \neg\chi'.$$

By propositional logic, $\Gamma^* \vDash \chi \lor \chi'$ and $\Delta^* \vDash \neg(\chi \lor \chi')$, so $\chi \lor \chi'$ separates $\Gamma^*$and Δ^*. A similar argument establishes that Δ^* is maximal.

Finally, we show that $\Gamma^* \cap \Delta^*$ is maximally consistent in $\mathcal{L}_0'$. It is obviously consistent, since it is the intersection of consistent sets. To show maximality, let $\sigma \in \mathcal{L}_0'$. Now, Γ^* is maximal in $\mathcal{L}_1' \supseteq \mathcal{L}_0'$, and similarly Δ^* is maximal in $\mathcal{L}_2' \supseteq \mathcal{L}_0'$. It follows that either $\sigma \in \Gamma^*$ or $\neg\sigma \in \Gamma^*$, and either $\sigma \in \Delta^*$ or $\neg\sigma \in \Delta^*$. If $\sigma \in \Gamma^*$ and $\neg\sigma \in \Delta^*$ then σ would separate Γ^* and Δ^*; and if $\neg\sigma \in \Gamma^*$ and $\sigma \in \Delta^*$ then Γ^* and Δ^* would be separated by $\neg\sigma$. Hence, either $\sigma \in \Gamma^* \cap \Delta^*$ or $\neg\sigma \in \Gamma^* \cap \Delta^*$, and $\Gamma^* \cap \Delta^*$ is maximal.

Since Γ^* is maximally consistent, it has a model $\mathfrak{M}_1'$ whose domain $|\mathfrak{M}_1'|$ comprises all and only the elements $c^{\mathfrak{M}_1'}$ interpreting the constant symbols—just like in the proof of the completeness theorem (??). Similarly, Δ^* has a model $\mathfrak{M}_2'$ whose domain $|\mathfrak{M}_2'|$ is given by the interpretations $c^{\mathfrak{M}_2'}$ of the constant symbols.

Let $\mathfrak{M}_1$ be obtained from $\mathfrak{M}_1'$ by dropping interpretations for constant symbols, function symbols, and predicate symbols in $\mathcal{L}_1' \setminus \mathcal{L}_0'$, and similarly for $\mathfrak{M}_2$. Then the map $h\colon M_1 \rightarrow M_2$ defined by $h(c^{\mathfrak{M}_1'}) = c^{\mathfrak{M}_2'}$ is an isomorphism in $\mathcal{L}_0'$, because $\Gamma^* \cap \Delta^*$ is maximally consistent in $\mathcal{L}_0'$, as shown. This follows because any $\mathcal{L}_0'$-sentence either belongs to both Γ^* and Δ^*, or to neither: so

$c^{\mathfrak{M}'_1} \in P^{\mathfrak{M}'_1}$ if and only if $P(c) \in \Gamma^*$ if and only if $P(c) \in \Delta^*$ if and only if $c^{\mathfrak{M}'_2} \in P^{\mathfrak{M}'_2}$. The other conditions satisfied by isomorphisms can be established similarly.

Let us now define a model $\mathfrak{M}$ for the language $\mathcal{L}_1 \cup \mathcal{L}_2$ as follows:

1. The domain $|\mathfrak{M}|$ is just $|\mathfrak{M}_2|$, i.e., the set of all elements $c^{\mathfrak{M}'_2}$;
2. If a predicate symbol P is in $\mathcal{L}_2 \setminus \mathcal{L}_1$ then $P^{\mathfrak{M}} = P^{\mathfrak{M}'_2}$;
3. If a predicate P is in $\mathcal{L}_1 \setminus \mathcal{L}_2$ then $P^{\mathfrak{M}} = h(P^{\mathfrak{M}'_2})$, i.e., $\langle c_1^{\mathfrak{M}'_2}, \dots, c_n^{\mathfrak{M}'_2} \rangle \in P^{\mathfrak{M}}$ if and only if $\langle c_1^{\mathfrak{M}'_1}, \dots, c_n^{\mathfrak{M}'_1} \rangle \in P^{\mathfrak{M}'_1}$.
4. If a predicate symbol P is in $\mathcal{L}_0$ then $P^{\mathfrak{M}} = P^{\mathfrak{M}'_2} = h(P^{\mathfrak{M}'_1})$.
5. Function symbols of $\mathcal{L}_1 \cup \mathcal{L}_2$, including constant symbols, are handled similarly.

Finally, one shows by induction on formulas that $\mathfrak{M}$ agrees with $\mathfrak{M}'_1$ on all formulas of $\mathcal{L}'_1$ and with $\mathfrak{M}'_2$ on all formulas of $\mathcal{L}'_2$. In particular, $\mathfrak{M} \vDash \Gamma^* \cup \Delta^*$, whence $\mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \vDash \neg\psi$, and $\nvDash \varphi \rightarrow \psi$. This concludes the proof of Craig's Interpolation Theorem. □

int.4 The Definability Theorem

mod:int:def:sec One important application of the interpolation theorem is Beth's definability theorem. To define an n-place relation R we can give a formula χ with n free variables which does not involve R. This would be an *explicit* definition of R in terms of χ. We can then say also that a theory $\Sigma(P)$ in a language containing the n-place predicate symbol P explicitly defines P if it contains (or at least entails) a formalized explicit definition, i.e.,

$$\Sigma(P) \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)).$$

But an explicit definition is only one way of defining—in the sense of determining completely—a relation. A theory may also be such that the interpretation of P is fixed by the interpretation of the rest of the language in any model. The definability theorem states that whenever a theory fixes the interpretation of P in this way—whenever it *implicitly defines* P—then it also explicitly defines it.

Definition int.5. Suppose $\mathcal{L}$ is a language not containing the predicate symbol P. A set $\Sigma(P)$ of sentences of $\mathcal{L} \cup \{P\}$ *explicitly defines* P if and only if there is a formula $\chi(x_1, \dots, x_n)$ of $\mathcal{L}$ such that

$$\Sigma(P) \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)).$$

Definition int.6. Suppose $\mathcal{L}$ is a language not containing the predicate symbols P and P'. A set $\Sigma(P)$ of sentences of $\mathcal{L} \cup \{P\}$ *implicitly defines* P if and only if

$$\Sigma(P) \cup \Sigma(P') \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow P'(x_1, \dots, x_n)),$$

where $\Sigma(P')$ is the result of uniformly replacing P with P' in $\Sigma(P)$.

In other words, for any model $\mathfrak{M}$ and $R, R' \subseteq |\mathfrak{M}|^n$, if both $(\mathfrak{M}, R) \vDash \Sigma(P)$ and $(\mathfrak{M}, R') \vDash \Sigma(P')$, then $R = R'$; where $(\mathfrak{M}, R)$ is the structure $\mathfrak{M}'$ for the expansion of $\mathcal{L}$ to $\mathcal{L} \cup \{P\}$ such that $P^{\mathfrak{M}'} = R$, and similarly for $(\mathfrak{M}, R')$.

Theorem int.7 (Beth Definability Theorem)**.** *A set $\Sigma(P)$ of $\mathcal{L} \cup \{P\}$-formulas implicitly defines P if and only $\Sigma(P)$ explicitly defines P.*

Proof. If $\Sigma(P)$ explicitly defines P then both

$$\begin{aligned} \Sigma(P) &\vDash \quad \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)) \\ \Sigma(P') &\vDash \quad \forall x_1 \dots \forall x_n \, (P'(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)) \end{aligned}$$

and the conclusion follows. For the converse: assume that $\Sigma(P)$ implicitly defines P. First, we add constant symbols $c_1, \dots, c_n$ to $\mathcal{L}$. Then

$$\Sigma(P) \cup \Sigma(P') \vDash P(c_1, \dots, c_n) \to P'(c_1, \dots, c_n).$$

By compactness, there are finite sets $\Delta_0 \subseteq \Sigma(P)$ and $\Delta_1 \subseteq \Sigma(P')$ such that

$$\Delta_0 \cup \Delta_1 \vDash P(c_1, \dots, c_n) \to P'(c_1, \dots, c_n).$$

Let $\theta(P)$ be the conjunction of all sentences $\varphi(P)$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$ and let $\theta(P')$ be the conjunction of all sentences $\varphi(P')$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$. Then $\theta(P) \wedge \theta(P') \vDash P(c_1, \dots, c_n) \to P'c_1 \dots c_n$. We can re-arrange this so that each predicate symbol occurs on one side of $\vDash$:

$$\theta(P) \wedge P(c_1, \dots, c_n) \vDash \theta(P') \to P'(c_1, \dots, c_n).$$

By Craig's Interpolation Theorem there is a sentence $\chi(c_1, \dots, c_n)$ not containing P or P' such that:

$$\theta(P) \wedge P(c_1, \dots, c_n) \vDash \chi(c_1, \dots, c_n); \quad \chi(c_1, \dots, c_n) \vDash \theta(P') \to P'(c_1, \dots, c_n).$$

From the former of these two entailments we have: $\theta(P) \vDash P(c_1, \dots, c_n) \to \chi(c_1, \dots, c_n)$. And from the latter, since an $\mathcal{L} \cup \{P\}$-model $(\mathfrak{M}, R) \vDash \varphi(P)$ if and only if the corresponding $\mathcal{L} \cup \{P'\}$-model $(\mathfrak{M}, R) \models \varphi(P')$, we have $\chi(c_1, \dots, c_n) \vDash \theta(P) \to P(c_1, \dots, c_n)$, from which:

$$\theta(P) \vDash \chi(c_1, \dots, c_n) \to P(c_1, \dots, c_n).$$

Putting the two together, $\theta(P) \vDash P(c_1, \dots, c_n) \leftrightarrow \chi(c_1, \dots, c_n)$, and by monotony and generalization also

$$\Sigma(P) \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)). \qquad \square$$

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Bibliography