int.1 The Definability Theorem

One important application of the interpolation theorem is Beth’s definability theorem. To define an $n$-place relation $R$ we can give a formula $\chi$ with $n$ free variables which does not involve $R$. This would be an explicit definition of $R$ in terms of $\chi$. We can then say also that a theory $\Sigma(P)$ in a language containing the $n$-place predicate symbol $P$ explicitly defines $P$ if it contains (or at least entails) a formalized explicit definition, i.e.,

$$\Sigma(P) \models \forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)).$$

But an explicit definition is only one way of defining—in the sense of determining completely—a relation. A theory may also be such that the interpretation of $P$ is fixed by the interpretation of the rest of the language in any model. The definability theorem states that whenever a theory fixes the interpretation of $P$ in this way—whenever it implicitly defines $P$—then it also explicitly defines it.

**Definition int.1.** Suppose $L$ is a language not containing the predicate symbol $P$. A set $\Sigma(P)$ of sentences of $L \cup \{P\}$ explicitly defines $P$ if and only if there is a formula $\chi(x_1, \ldots, x_n)$ of $L$ such that

$$\Sigma(P) \models \forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)).$$

**Definition int.2.** Suppose $L$ is a language not containing the predicate symbols $P$ and $P'$. A set $\Sigma(P)$ of sentences of $L \cup \{P\}$ implicitly defines $P$ if and only if

$$\Sigma(P) \cup \Sigma(P') \models \forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_n) \leftrightarrow P'(x_1, \ldots, x_n)),$$

where $\Sigma(P')$ is the result of uniformly replacing $P$ with $P'$ in $\Sigma(P)$.

In other words, for any model $\mathfrak{M}$ and $R, R' \subseteq [\mathfrak{M}]^n$, if both $(\mathfrak{M}, R) \models \Sigma(P)$ and $(\mathfrak{M}, R') \models \Sigma(P')$, then $R = R'$; where $(\mathfrak{M}, R)$ is the structure $\mathfrak{M}$ for the expansion of $L$ to $L \cup \{P\}$ such that $P_{\mathfrak{M}^R} = R$, and similarly for $(\mathfrak{M}, R').$

**Theorem int.3** (Beth Definability Theorem). A set $\Sigma(P)$ of $L \cup \{P\}$-formulas implicitly defines $P$ if and only $\Sigma(P)$ explicitly defines $P$.

**Proof.** If $\Sigma(P)$ explicitly defines $P$ then both

$$\Sigma(P) \models \forall x_1 \ldots \forall x_n [(P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n))]
\Sigma(P') \models \forall x_1 \ldots \forall x_n [(P'(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n))]$$

and the conclusion follows. For the converse: assume that $\Sigma(P)$ implicitly defines $P$. First, we add constant symbols $c_1, \ldots, c_n$ to $L$. Then

$$\Sigma(P) \cup \Sigma(P') \models P(c_1, \ldots, c_n) \rightarrow P'(c_1, \ldots, c_n).$$

By compactness, there are finite sets $\Delta_0 \subseteq \Sigma(P)$ and $\Delta_1 \subseteq \Sigma(P')$ such that

$$\Delta_0 \cup \Delta_1 \models P(c_1, \ldots, c_n) \rightarrow P'(c_1, \ldots, c_n).$$
Let $\theta(P)$ be the conjunction of all sentences $\varphi(P)$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$ and let $\theta(P')$ be the conjunction of all sentences $\varphi(P')$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$. Then $\theta(P) \land \theta(P') \models P(c_1, \ldots, c_n) \rightarrow P'c_1 \ldots c_n$. We can re-arrange this so that each predicate symbol occurs on one side of $\models$:

$$\theta(P) \land P(c_1, \ldots, c_n) \models \theta(P') \rightarrow P'(c_1, \ldots, c_n).$$

By Craig’s Interpolation Theorem there is a sentence $\chi(c_1, \ldots, c_n)$ not containing $P$ or $P'$ such that:

$$\theta(P) \land P(c_1, \ldots, c_n) \models \chi(c_1, \ldots, c_n); \quad \chi(c_1, \ldots, c_n) \models \theta(P') \rightarrow P'(c_1, \ldots, c_n).$$

From the former of these two entailments we have: $\theta(P) \models P(c_1, \ldots, c_n) \rightarrow \chi(c_1, \ldots, c_n)$. And from the latter, since an $L \cup \{P\}$-model $(M, R) \models \varphi(P)$ if and only if the corresponding $L \cup \{P'\}$-model $(M, R) \models \varphi(P')$, we have $\chi(c_1, \ldots, c_n) \models \theta(P) \rightarrow P(c_1, \ldots, c_n)$, from which:

$$\theta(P) \models \chi(c_1, \ldots, c_n) \rightarrow P(c_1, \ldots, c_n).$$

Putting the two together, $\theta(P) \models P(c_1, \ldots, c_n) \leftrightarrow \chi(c_1, \ldots, c_n)$, and by monotony and generalization also

$$\Sigma(P) \models \forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_n)).$$

\[ \square \]

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**Bibliography**