bas.1 The Theory of a Structure

Every structure \(M\) makes some sentences true, and some false. The set of all the sentences it makes true is called its \(theory\). That set is in fact a theory, since anything it entails must be true in all its models, including \(M\).

Definition bas.1. Given a structure \(M\), the \theory of \(M\) is the set \(\text{Th}(M)\) of sentences that are true in \(M\), i.e., \(\text{Th}(M) = \{ \varphi : M \models \varphi \}\).

We also use the term “theory” informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

Proposition bas.2. For any \(M\), \(\text{Th}(M)\) is complete.

Proof. For any sentence \(\varphi\) either \(M \models \varphi\) or \(M \models \lnot \varphi\), so either \(\varphi \in \text{Th}(M)\) or \(\lnot \varphi \in \text{Th}(M)\). \(\square\)

Proposition bas.3. If \(N \models \varphi\) for every \(\varphi \in \text{Th}(M)\), then \(M \equiv N\).

Proof. Since \(N \models \varphi\) for all \(\varphi \in \text{Th}(M)\), \(\text{Th}(M) \subseteq \text{Th}(N)\). If \(N \models \varphi\), then \(N \not\models \lnot \varphi\), so \(\lnot \varphi \notin \text{Th}(N)\). Since \(\text{Th}(M)\) is complete, \(\varphi \in \text{Th}(M)\). So, \(\text{Th}(M) \subseteq \text{Th}(N)\), and we have \(M \equiv N\). \(\square\)

Remark 1. Consider \(\mathcal{R} = (\mathbb{R}, <)\), the structure whose domain is the set \(\mathbb{R}\) of the real numbers, in the language comprising only a 2-place predicate symbol interpreted as the \(<\) relation over the reals. Clearly \(\mathcal{R}\) is non-enumerable; however, since \(\text{Th}(\mathcal{R})\) is obviously consistent, by the Löwenheim-Skolem theorem it has an enumerable model, say \(\mathcal{S}\), and by Proposition bas.3, \(\mathcal{R} \equiv \mathcal{S}\). Moreover, since \(\mathcal{R}\) and \(\mathcal{S}\) are not isomorphic, this shows that the converse of ?? fails in general.

Photo Credits

Bibliography