

## bas.1 The Theory of a Structure

Every structure  $\mathfrak{M}$  makes some sentences true, and some false. The set of all the sentences it makes true is called its *theory*. That set is in fact a theory, since anything it entails must be true in all its models, including  $\mathfrak{M}$ .

**Definition bas.1.** Given a structure  $\mathfrak{M}$ , the *theory* of  $\mathfrak{M}$  is the set  $\text{Th}(\mathfrak{M})$  of sentences that are true in  $\mathfrak{M}$ , i.e.,  $\text{Th}(\mathfrak{M}) = \{\varphi : \mathfrak{M} \models \varphi\}$ .

We also use the term “theory” informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

**Proposition bas.2.** For any  $\mathfrak{M}$ ,  $\text{Th}(\mathfrak{M})$  is complete.

*Proof.* For any sentence  $\varphi$  either  $\mathfrak{M} \models \varphi$  or  $\mathfrak{M} \models \neg\varphi$ , so either  $\varphi \in \text{Th}(\mathfrak{M})$  or  $\neg\varphi \in \text{Th}(\mathfrak{M})$ .  $\square$

mod:bas:thm:  
prop:equiv

**Proposition bas.3.** If  $\mathfrak{N} \models \varphi$  for every  $\varphi \in \text{Th}(\mathfrak{M})$ , then  $\mathfrak{M} \equiv \mathfrak{N}$ .

*Proof.* Since  $\mathfrak{N} \models \varphi$  for all  $\varphi \in \text{Th}(\mathfrak{M})$ ,  $\text{Th}(\mathfrak{M}) \subseteq \text{Th}(\mathfrak{N})$ . If  $\mathfrak{N} \models \varphi$ , then  $\mathfrak{N} \not\models \neg\varphi$ , so  $\neg\varphi \notin \text{Th}(\mathfrak{M})$ . Since  $\text{Th}(\mathfrak{M})$  is complete,  $\varphi \in \text{Th}(\mathfrak{M})$ . So,  $\text{Th}(\mathfrak{N}) \subseteq \text{Th}(\mathfrak{M})$ , and we have  $\mathfrak{M} \equiv \mathfrak{N}$ .  $\square$

mod:bas:thm:  
remark:R

*Remark 1.* Consider  $\mathfrak{R} = \langle \mathbb{R}, < \rangle$ , the structure whose domain is the set  $\mathbb{R}$  of the real numbers, in the language comprising only a 2-place predicate symbol interpreted as the  $<$  relation over the reals. Clearly  $\mathfrak{R}$  is non-enumerable; however, since  $\text{Th}(\mathfrak{R})$  is obviously consistent, by the Löwenheim–Skolem theorem it has an enumerable model, say  $\mathfrak{S}$ , and by Proposition bas.3,  $\mathfrak{R} \equiv \mathfrak{S}$ . Moreover, since  $\mathfrak{R}$  and  $\mathfrak{S}$  are not isomorphic, this shows that the converse of ?? fails in general.

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## Bibliography