bas.1  The Theory of a Structure

Every structure $\mathcal{M}$ makes some sentences true, and some false. The set of all the sentences it makes true is called its theory. That set is in fact a theory, since anything it entails must be true in all its models, including $\mathcal{M}$.

Definition bas.1. Given a structure $\mathcal{M}$, the theory of $\mathcal{M}$ is the set $\text{Th}(\mathcal{M})$ of sentences that are true in $\mathcal{M}$, i.e., $\text{Th}(\mathcal{M}) = \{ \varphi : \mathcal{M} \models \varphi \}$.

We also use the term “theory” informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

Proposition bas.2. For any $\mathcal{M}$, $\text{Th}(\mathcal{M})$ is complete.

Proof. For any sentence $\varphi$ either $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg \varphi$, so either $\varphi \in \text{Th}(\mathcal{M})$ or $\neg \varphi \in \text{Th}(\mathcal{M})$. □

Proposition bas.3. If $\mathcal{N} \models \varphi$ for every $\varphi \in \text{Th}(\mathcal{M})$, then $\mathcal{M} \equiv \mathcal{N}$.

Proof. Since $\mathcal{N} \models \varphi$ for all $\varphi \in \text{Th}(\mathcal{M})$, $\text{Th}(\mathcal{M}) \subseteq \text{Th}(\mathcal{N})$. If $\mathcal{N} \models \varphi$, then $\mathcal{N} \models \neg \varphi$, so $\neg \varphi \not\in \text{Th}(\mathcal{M})$. Since $\text{Th}(\mathcal{M})$ is complete, $\varphi \in \text{Th}(\mathcal{M})$. So, $\text{Th}(\mathcal{N}) \subseteq \text{Th}(\mathcal{M})$, and we have $\mathcal{M} \equiv \mathcal{N}$. □

Remark 1. Consider $\mathcal{R} = (\mathbb{R}, <)$, the structure whose domain is the set $\mathbb{R}$ of the real numbers, in the language comprising only a 2-place predicate symbol interpreted as the $<$ relation over the reals. Clearly $\mathcal{R}$ is non-enumerable; however, since $\text{Th}(\mathcal{R})$ is obviously consistent, by the Löwenheim-Skolem theorem it has an enumerable model, say $\mathcal{S}$, and by Proposition bas.3, $\mathcal{R} \equiv \mathcal{S}$. Moreover, since $\mathcal{R}$ and $\mathcal{S}$ are not isomorphic, this shows that the converse of ?? fails in general.

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Bibliography