

bas.1 The Theory of a Structure

Every structure \mathfrak{M} makes some sentences true, and some false. The set of all the sentences it makes true is called its *theory*. That set is in fact a theory, since anything it entails must be true in all its models, including \mathfrak{M} .

Definition bas.1. Given a structure \mathfrak{M} , the *theory* of \mathfrak{M} is the set $\text{Th}(\mathfrak{M})$ of sentences that are true in \mathfrak{M} , i.e., $\text{Th}(\mathfrak{M}) = \{\varphi : \mathfrak{M} \models \varphi\}$.

We also use the term “theory” informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

Proposition bas.2. For any \mathfrak{M} , $\text{Th}(\mathfrak{M})$ is complete.

Proof. For any sentence φ either $\mathfrak{M} \models \varphi$ or $\mathfrak{M} \models \neg\varphi$, so either $\varphi \in \text{Th}(\mathfrak{M})$ or $\neg\varphi \in \text{Th}(\mathfrak{M})$. \square

mod:bas:thm:
prop:equiv **Proposition bas.3.** If $\mathfrak{N} \models \varphi$ for every $\varphi \in \text{Th}(\mathfrak{M})$, then $\mathfrak{M} \equiv \mathfrak{N}$.

Proof. Since $\mathfrak{N} \models \varphi$ for all $\varphi \in \text{Th}(\mathfrak{M})$, $\text{Th}(\mathfrak{M}) \subseteq \text{Th}(\mathfrak{N})$. If $\mathfrak{N} \models \varphi$, then $\mathfrak{N} \not\models \neg\varphi$, so $\neg\varphi \notin \text{Th}(\mathfrak{M})$. Since $\text{Th}(\mathfrak{M})$ is complete, $\varphi \in \text{Th}(\mathfrak{M})$. So, $\text{Th}(\mathfrak{N}) \subseteq \text{Th}(\mathfrak{M})$, and we have $\mathfrak{M} \equiv \mathfrak{N}$. \square

mod:bas:thm:
remark:R *Remark 1.* Consider $\mathfrak{R} = \langle \mathbb{R}, < \rangle$, the structure whose domain is the set \mathbb{R} of the real numbers, in the language comprising only a 2-place predicate symbol interpreted as the $<$ relation over the reals. Clearly \mathfrak{R} is non-enumerable; however, since $\text{Th}(\mathfrak{R})$ is obviously consistent, by the Löwenheim-Skolem theorem it has an enumerable model, say \mathfrak{S} , and by Proposition bas.3, $\mathfrak{R} \equiv \mathfrak{S}$. Moreover, since \mathfrak{R} and \mathfrak{S} are not isomorphic, this shows that the converse of ?? fails in general.

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Bibliography