

bas.1 Partial Isomorphisms

Definition bas.1. Given two **structures** \mathfrak{M} and \mathfrak{N} , a *partial isomorphism* from \mathfrak{M} to \mathfrak{N} is a finite partial function p taking arguments in $|\mathfrak{M}|$ and returning values in $|\mathfrak{N}|$, which satisfies the isomorphism conditions from ?? on its domain:

1. p is **injective**;
2. for every **constant symbol** c : if $p(c^{\mathfrak{M}})$ is defined, then $p(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$;
3. for every n -place **predicate symbol** P : if a_1, \dots, a_n are in the domain of p , then $\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}}$ if and only if $\langle p(a_1), \dots, p(a_n) \rangle \in P^{\mathfrak{N}}$;
4. for every n -place **function symbol** f : if a_1, \dots, a_n are in the domain of p , then $p(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{N}}(p(a_1), \dots, p(a_n))$.

That p is finite means that $\text{dom}(p)$ is finite.

Notice that the empty function \emptyset is always a partial isomorphism between any two **structures**.

mod:bas:pis:
defn:partialisom

Definition bas.2. Two **structures** \mathfrak{M} and \mathfrak{N} , are *partially isomorphic*, written $\mathfrak{M} \simeq_p \mathfrak{N}$, if and only if there is a non-empty set I of partial isomorphisms between \mathfrak{M} and \mathfrak{N} satisfying the *back-and-forth* property:

1. (*Forth*) For every $p \in I$ and $a \in |\mathfrak{M}|$ there is $q \in I$ such that $p \subseteq q$ and a is in the domain of q ;
2. (*Back*) For every $p \in I$ and $b \in |\mathfrak{N}|$ there is $q \in I$ such that $p \subseteq q$ and b is in the range of q .

mod:bas:pis:
thm:p-isom1

Theorem bas.3. If $\mathfrak{M} \simeq_p \mathfrak{N}$ and \mathfrak{M} and \mathfrak{N} are **enumerable**, then $\mathfrak{M} \simeq \mathfrak{N}$.

Proof. Since \mathfrak{M} and \mathfrak{N} are **enumerable**, let $|\mathfrak{M}| = \{a_0, a_1, \dots\}$ and $|\mathfrak{N}| = \{b_0, b_1, \dots\}$. Starting with an arbitrary $p_0 \in I$, we define an increasing sequence of partial isomorphisms $p_0 \subseteq p_1 \subseteq p_2 \subseteq \dots$ as follows:

1. if $n + 1$ is odd, say $n = 2r$, then using the Forth property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and a_r is in the domain of p_{n+1} ;
2. if $n + 1$ is even, say $n + 1 = 2r$, then using the Back property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and b_r is in the range of p_{n+1} .

If we now put:

$$p = \bigcup_{n \geq 0} p_n,$$

we have that p is an isomorphism between \mathfrak{M} and \mathfrak{N} . □

Problem bas.1. Show in detail that p as defined in [Theorem bas.3](#) is in fact an isomorphism.

Theorem bas.4. Suppose \mathfrak{M} and \mathfrak{N} are *structures* for a purely relational language (a language containing only predicate symbols, and no function symbols or constants). Then if $\mathfrak{M} \simeq_p \mathfrak{N}$, also $\mathfrak{M} \equiv \mathfrak{N}$.

*mod:bas:pis:
thm:p-isom2*

Proof. By induction on formulas, one shows that if a_1, \dots, a_n and b_1, \dots, b_n are such that there is a partial isomorphism p mapping each a_i to b_i and $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ (for $i = 1, \dots, n$), then $\mathfrak{M}, s_1 \models \varphi$ if and only if $\mathfrak{N}, s_2 \models \varphi$. The case for $n = 0$ gives $\mathfrak{M} \equiv \mathfrak{N}$. \square

Remark 1. If function symbols are present, the previous result is still true, but one needs to consider the isomorphism induced by p between the substructure of \mathfrak{M} generated by a_1, \dots, a_n and the substructure of \mathfrak{N} generated by b_1, \dots, b_n .

The previous result can be “broken down” into stages by establishing a connection between the number of nested quantifiers in a formula and how many times the relevant partial isomorphisms can be extended.

Definition bas.5. For any formula φ , the *quantifier rank* of φ , denoted by $\text{qr}(\varphi) \in \mathbb{N}$, is recursively defined as the highest number of nested quantifiers in φ . Two structures \mathfrak{M} and \mathfrak{N} are *n-equivalent*, written $\mathfrak{M} \equiv_n \mathfrak{N}$, if they agree on all sentences of quantifier rank less than or equal to n .

Proposition bas.6. Let \mathcal{L} be a finite purely relational language, i.e., a language containing finitely many predicate symbols and constant symbols, and no function symbols. Then for each $n \in \mathbb{N}$ there are only finitely many first-order sentences in the language \mathcal{L} that have quantifier rank no greater than n , up to logical equivalence.

*mod:bas:pis:
prop:qr-finite*

Proof. By induction on n . \square

Definition bas.7. Given a structure \mathfrak{M} , let $|\mathfrak{M}|^{<\omega}$ be the set of all finite sequences over $|\mathfrak{M}|$. We use $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ to range over finite sequences of elements. If $\mathbf{a} \in |\mathfrak{M}|^{<\omega}$ and $a \in |\mathfrak{M}|$, then $\mathbf{a}a$ represents the *concatenation* of \mathbf{a} with a .

Definition bas.8. Given structures \mathfrak{M} and \mathfrak{N} , we define relations $I_n \subseteq |\mathfrak{M}|^{<\omega} \times |\mathfrak{N}|^{<\omega}$ between sequences of equal length, by recursion on n as follows:

1. $I_0(\mathbf{a}, \mathbf{b})$ if and only if \mathbf{a} and \mathbf{b} satisfy the same atomic formulas in \mathfrak{M} and \mathfrak{N} ; i.e., if $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ and φ is atomic with all variables among x_1, \dots, x_n , then $\mathfrak{M}, s_1 \models \varphi$ if and only if $\mathfrak{N}, s_2 \models \varphi$.
2. $I_{n+1}(\mathbf{a}, \mathbf{b})$ if and only if for every $a \in A$ there is a $b \in B$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and vice-versa.

Definition bas.9. Write $\mathfrak{M} \approx_n \mathfrak{N}$ if $I_n(\emptyset, \emptyset)$ holds of \mathfrak{M} and \mathfrak{N} (where \emptyset is the empty sequence).

mod:bas:pis:
thm:b-n-f

Theorem bas.10. *Let \mathcal{L} be a purely relational language. Then $I_n(\mathbf{a}, \mathbf{b})$ implies that for every φ such that $\text{qr}(\varphi) \leq n$, we have $\mathfrak{M}, \mathbf{a} \models \varphi$ if and only if $\mathfrak{N}, \mathbf{b} \models \varphi$ (where again \mathbf{a} satisfies φ if any s such that $s(x_i) = a_i$ satisfies φ). Moreover, if \mathcal{L} is finite, the converse also holds.*

Proof. The proof that $I_n(\mathbf{a}, \mathbf{b})$ implies that \mathbf{a} and \mathbf{b} satisfy the same formulas of quantifier rank no greater than n is by an easy induction on φ . For the converse we proceed by induction on n , using Proposition bas.6, which ensures that for each n there are at most finitely many non-equivalent formulas of that quantifier rank.

For $n = 0$ the hypothesis that \mathbf{a} and \mathbf{b} satisfy the same quantifier-free formulas gives that they satisfy the same atomic ones, so that $I_0(\mathbf{a}, \mathbf{b})$.

For the $n + 1$ case, suppose that \mathbf{a} and \mathbf{b} satisfy the same formulas of quantifier rank no greater than $n + 1$; in order to show that $I_{n+1}(\mathbf{a}, \mathbf{b})$ suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and by the inductive hypothesis again suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n .

Given $a \in |\mathfrak{M}|$, let τ_n^a be set of formulas $\psi(x, \mathbf{y})$ of rank no greater than n satisfied by $\mathbf{a}a$ in \mathfrak{M} ; τ_n^a is finite, so we can assume it is a single first-order formula. It follows that \mathbf{a} satisfies $\exists x \tau_n^a(x, \mathbf{y})$, which has quantifier rank no greater than $n + 1$. By hypothesis \mathbf{b} satisfies the same formula in \mathfrak{N} , so that there is a $b \in |\mathfrak{N}|$ such that $\mathbf{b}b$ satisfies τ_n^a ; in particular, $\mathbf{b}b$ satisfies the same formulas of quantifier rank no greater than n as $\mathbf{a}a$. Similarly one shows that for every $b \in |\mathfrak{N}|$ there is $a \in |\mathfrak{M}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n , which completes the proof. \square

mod:bas:pis:
cor:b-n-f

Corollary bas.11. *If \mathfrak{M} and \mathfrak{N} are purely relational structures in a finite language, then $\mathfrak{M} \approx_n \mathfrak{N}$ if and only if $\mathfrak{M} \equiv_n \mathfrak{N}$. In particular $\mathfrak{M} \equiv \mathfrak{N}$ if and only if for each n , $\mathfrak{M} \approx_n \mathfrak{N}$.*

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Bibliography