bas.1 Partial Isomorphisms

Definition bas.1. Given two structures \( \mathcal{M} \) and \( \mathcal{N} \), a partial isomorphism from \( \mathcal{M} \) to \( \mathcal{N} \) is a finite partial function \( p \) taking arguments in \( |\mathcal{M}| \) and returning values in \( |\mathcal{N}| \), which satisfies the isomorphism conditions from ?? on its domain:

1. \( p \) is injective;
2. for every constant symbol \( c \): if \( p(c^\mathcal{M}) \) is defined, then \( p(c^\mathcal{M}) = c^\mathcal{N} \);
3. for every \( n \)-place predicate symbol \( P \): if \( a_1, \ldots, a_n \) are in the domain of \( p \), then \( \langle a_1, \ldots, a_n \rangle \in P^\mathcal{M} \) if and only if \( \langle p(a_1), \ldots, p(a_n) \rangle \in P^\mathcal{N} \);
4. for every \( n \)-place function symbol \( f \): if \( a_1, \ldots, a_n \) are in the domain of \( p \), then \( p(f^\mathcal{M}(a_1, \ldots, a_n)) = f^\mathcal{N}(p(a_1), \ldots, p(a_n)) \).

That \( p \) is finite means that \( \text{dom}(p) \) is finite.

Notice that the empty function \( \emptyset \) is always a partial isomorphism between any two structures.

Definition bas.2. Two structures \( \mathcal{M} \) and \( \mathcal{N} \), are partially isomorphic, written \( \mathcal{M} \simeq_p \mathcal{N} \), if and only if there is a non-empty set \( I \) of partial isomorphisms between \( \mathcal{M} \) and \( \mathcal{N} \) satisfying the back-and-forth property:

1. (Forth) For every \( p \in I \) and \( a \in |\mathcal{M}| \) there is \( q \in I \) such that \( p \subseteq q \) and \( a \) is in the domain of \( q \);
2. (Back) For every \( p \in I \) and \( b \in |\mathcal{N}| \) there is \( q \in I \) such that \( p \subseteq q \) and \( b \) is in the range of \( q \).

Theorem bas.3. If \( \mathcal{M} \simeq_p \mathcal{N} \) and \( \mathcal{M} \) and \( \mathcal{N} \) are enumerable, then \( \mathcal{M} \simeq \mathcal{N} \).

Proof. Since \( \mathcal{M} \) and \( \mathcal{N} \) are enumerable, let \( |\mathcal{M}| = \{a_0, a_1, \ldots \} \) and \( |\mathcal{N}| = \{b_0, b_1, \ldots \} \). Starting with an arbitrary \( p_0 \in I \), we define an increasing sequence of partial isomorphisms \( p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots \) as follows:

1. if \( n + 1 \) is odd, say \( n = 2r \), then using the Forth property find a \( p_{n+1} \in I \) such that \( p_n \subseteq p_{n+1} \) and \( a_r \) is in the domain of \( p_{n+1} \);
2. if \( n + 1 \) is even, say \( n + 1 = 2r \), then using the Back property find a \( p_{n+1} \in I \) such that \( p_n \subseteq p_{n+1} \) and \( b_r \) is in the range of \( p_{n+1} \).

If we now put:

\[
p = \bigcup_{n \geq 0} p_n,
\]

we have that \( p \) is a an isomorphism between \( \mathcal{M} \) and \( \mathcal{N} \). \( \square \)

Problem bas.1. Show in detail that \( p \) as defined in Theorem bas.3 is in fact an isomorphism.
Theorem bas.4. Suppose $\mathcal{M}$ and $\mathcal{N}$ are structures for a purely relational language (a language containing only predicate symbols, and no function symbols or constants). Then if $\mathcal{M} \simeq_p \mathcal{N}$, also $\mathcal{M} \equiv \mathcal{N}$.

Proof. By induction on formulas, one shows that if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are such that there is a partial isomorphism $p$ mapping each $a_i$ to $b_i$ and $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ (for $i = 1, \ldots, n$), then $\mathcal{M}, s_1 \models \varphi$ if and only if $\mathcal{N}, s_2 \models \varphi$. The case for $n = 0$ gives $\mathcal{M} \equiv \mathcal{N}$.

Remark 1. If function symbols are present, the previous result is still true, but one needs to consider the isomorphism induced by $p$ between the substructure of $\mathcal{M}$ generated by $a_1, \ldots, a_n$ and the substructure of $\mathcal{N}$ generated by $b_1, \ldots, b_n$.

The previous result can be “broken down” into stages by establishing a connection between the number of nested quantifiers in a formula and how many times the relevant partial isomorphisms can be extended.

Definition bas.5. For any formula $\varphi$, the quantifier rank of $\varphi$, denoted by $\text{qr}(\varphi) \in \mathbb{N}$, is recursively defined as the highest number of nested quantifiers in $\varphi$. Two structures $\mathcal{M}$ and $\mathcal{N}$ are $n$-equivalent, written $\mathcal{M} \equiv_n \mathcal{N}$, if they agree on all sentences of quantifier rank less than or equal to $n$.

Proposition bas.6. Let $\mathcal{L}$ be a finite purely relational language, i.e., a language containing finitely many predicate symbols and constant symbols, and no function symbols. Then for each $n \in \mathbb{N}$ there are only finitely many first-order sentences in the language $\mathcal{L}$ that have quantifier rank no greater than $n$, up to logical equivalence.

Definition bas.7. Given a structure $\mathcal{M}$, let $|\mathcal{M}|^{<\omega}$ be the set of all finite sequences over $|\mathcal{M}|$. We use $a, b, c, \ldots$ to range over finite sequences of elements. If $a \in |\mathcal{M}|^{<\omega}$ and $\sigma \in |\mathcal{M}|$, then $a\sigma$ represents the concatenation of $a$ with $\sigma$.

Definition bas.8. Given structures $\mathcal{M}$ and $\mathcal{N}$, we define relations $I_n \subseteq |\mathcal{M}|^{<\omega} \times |\mathcal{N}|^{<\omega}$ between sequences of equal length, by recursion on $n$ as follows:

1. $I_0(a, b)$ if and only if $a$ and $b$ satisfy the same atomic formulas in $\mathcal{M}$ and $\mathcal{N}$; i.e., if $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ and $\varphi$ is atomic with all variables among $x_1, \ldots, x_n$, then $\mathcal{M}, s_1 \models \varphi$ if and only if $\mathcal{N}, s_2 \models \varphi$.

2. $I_{n+1}(a, b)$ if and only if for every $a \in A$ there is a $b \in B$ such that $I_n(a\sigma, b\sigma)$, and vice-versa.

Definition bas.9. Write $\mathcal{M} \approx_n \mathcal{N}$ if $I_n(A, A)$ holds of $\mathcal{M}$ and $\mathcal{N}$ (where $A$ is the empty sequence).
Theorem bas.10. Let \( \mathcal{L} \) be a purely relational language. Then \( I_n(a, b) \) implies that for every \( \varphi \) such that \( qr(\varphi) \leq n \), we have \( \mathcal{M}, a \models \varphi \) if and only if \( \mathcal{N}, b \models \varphi \) (where again \( a \) satisfies \( \varphi \) if any \( s \) such that \( s(x_i) = a_i \) satisfies \( \varphi \)). Moreover, if \( \mathcal{L} \) is finite, the converse also holds.

Proof. The proof that \( I_n(a, b) \) implies that \( a \) and \( b \) satisfy the same formulas of quantifier rank no greater than \( n \) is by an easy induction on \( \varphi \). For the converse we proceed by induction on \( n \), using Proposition bas.6, which ensures that for each \( n \) there are at most finitely many non-equivalent formulas of that quantifier rank.

For \( n = 0 \) the hypothesis that \( a \) and \( b \) satisfy the same quantifier-free formulas gives that they satisfy the same atomic ones, so that \( I_0(a, b) \).

For the \( n + 1 \) case, suppose that \( a \) and \( b \) satisfy the same formulas of quantifier rank no greater than \( n + 1 \); in order to show that \( I_{n+1}(a, b) \) suffices to show that for each \( a \in |\mathcal{M}| \) there is a \( b \in |\mathcal{N}| \) such that \( I_n(aa, bb) \), and by the inductive hypothesis again suffices to show that for each \( a \in |\mathcal{M}| \) there is a \( b \in |\mathcal{N}| \) such that \( aa \) and \( bb \) satisfy the same formulas of quantifier rank no greater than \( n \).

Given \( a \in |\mathcal{M}| \), let \( \tau_n^a \) be set of formulas \( \psi(x, y) \) of rank no greater than \( n \) satisfied by \( aa \) in \( \mathcal{M} \); \( \tau_n^a \) is finite, so we can assume it is a single first-order formula. It follows that \( a \) satisfies \( \exists x \tau_n^a(x, y) \), which has quantifier rank no greater than \( n + 1 \). By hypothesis \( b \) satisfies the same formula in \( \mathcal{N} \), so that there is a \( b \in |\mathcal{N}| \) such that \( bb \) satisfies \( \tau_n^a \); in particular, \( bb \) satisfies the same formulas of quantifier rank no greater than \( n \) as \( aa \). Similarly one shows that for every \( b \in |\mathcal{N}| \) there is \( a \in |\mathcal{M}| \) such that \( aa \) and \( bb \) satisfy the same formulas of quantifier rank no greater than \( n \), which completes the proof.

Corollary bas.11. If \( \mathcal{M} \) and \( \mathcal{N} \) are purely relational structures in a finite language, then \( \mathcal{M} \equiv_n \mathcal{N} \) if and only if \( \mathcal{M} \equiv \mathcal{N} \). In particular \( \mathcal{M} \equiv \mathcal{N} \) if and only if for each \( n \), \( \mathcal{M} \equiv_n \mathcal{N} \).

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Bibliography