

## bas.1 Partial Isomorphisms

**Definition bas.1.** Given two **structures**  $\mathfrak{M}$  and  $\mathfrak{N}$ , a *partial isomorphism* from  $\mathfrak{M}$  to  $\mathfrak{N}$  is a finite partial function  $p$  taking arguments in  $|\mathfrak{M}|$  and returning values in  $|\mathfrak{N}|$ , which satisfies the isomorphism conditions from ?? on its domain:

1.  $p$  is **injective**;
2. for every **constant symbol**  $c$ : if  $p(c^{\mathfrak{M}})$  is defined, then  $p(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$ ;
3. for every  $n$ -place **predicate symbol**  $P$ : if  $a_1, \dots, a_n$  are in the domain of  $p$ , then  $\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}}$  if and only if  $\langle p(a_1), \dots, p(a_n) \rangle \in P^{\mathfrak{N}}$ ;
4. for every  $n$ -place **function symbol**  $f$ : if  $a_1, \dots, a_n$  are in the domain of  $p$ , then  $p(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{N}}(p(a_1), \dots, p(a_n))$ .

That  $p$  is finite means that  $\text{dom}(p)$  is finite.

Notice that the empty function  $\emptyset$  is always a partial isomorphism between any two **structures**.

mod:bas:pis:  
defn:partialisom

**Definition bas.2.** Two **structures**  $\mathfrak{M}$  and  $\mathfrak{N}$ , are *partially isomorphic*, written  $\mathfrak{M} \simeq_p \mathfrak{N}$ , if and only if there is a non-empty set  $I$  of partial isomorphisms between  $\mathfrak{M}$  and  $\mathfrak{N}$  satisfying the *back-and-forth* property:

1. (*Forth*) For every  $p \in I$  and  $a \in |\mathfrak{M}|$  there is  $q \in I$  such that  $p \subseteq q$  and  $a$  is in the domain of  $q$ ;
2. (*Back*) For every  $p \in I$  and  $b \in |\mathfrak{N}|$  there is  $q \in I$  such that  $p \subseteq q$  and  $b$  is in the range of  $q$ .

mod:bas:pis:  
thm:p-isom1

**Theorem bas.3.** If  $\mathfrak{M} \simeq_p \mathfrak{N}$  and  $\mathfrak{M}$  and  $\mathfrak{N}$  are **enumerable**, then  $\mathfrak{M} \simeq \mathfrak{N}$ .

*Proof.* Since  $\mathfrak{M}$  and  $\mathfrak{N}$  are **enumerable**, let  $|\mathfrak{M}| = \{a_0, a_1, \dots\}$  and  $|\mathfrak{N}| = \{b_0, b_1, \dots\}$ . Starting with an arbitrary  $p_0 \in I$ , we define an increasing sequence of partial isomorphisms  $p_0 \subseteq p_1 \subseteq p_2 \subseteq \dots$  as follows:

1. if  $n + 1$  is odd, say  $n = 2r$ , then using the Forth property find a  $p_{n+1} \in I$  such that  $p_n \subseteq p_{n+1}$  and  $a_r$  is in the domain of  $p_{n+1}$ ;
2. if  $n + 1$  is even, say  $n + 1 = 2r$ , then using the Back property find a  $p_{n+1} \in I$  such that  $p_n \subseteq p_{n+1}$  and  $b_r$  is in the range of  $p_{n+1}$ .

If we now put:

$$p = \bigcup_{n \geq 0} p_n,$$

we have that  $p$  is an isomorphism between  $\mathfrak{M}$  and  $\mathfrak{N}$ . □

**Problem bas.1.** Show in detail that  $p$  as defined in [Theorem bas.3](#) is in fact an isomorphism.

**Theorem bas.4.** Suppose  $\mathfrak{M}$  and  $\mathfrak{N}$  are *structures* for a purely relational language (a language containing only predicate symbols, and no function symbols or constants). Then if  $\mathfrak{M} \simeq_p \mathfrak{N}$ , also  $\mathfrak{M} \equiv \mathfrak{N}$ .

*mod:bas:pis:  
thm:p-isom2*

*Proof.* By induction on formulas, one shows that if  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  are such that there is a partial isomorphism  $p$  mapping each  $a_i$  to  $b_i$  and  $s_1(x_i) = a_i$  and  $s_2(x_i) = b_i$  (for  $i = 1, \dots, n$ ), then  $\mathfrak{M}, s_1 \models \varphi$  if and only if  $\mathfrak{N}, s_2 \models \varphi$ . The case for  $n = 0$  gives  $\mathfrak{M} \equiv \mathfrak{N}$ .  $\square$

*Remark 1.* If function symbols are present, the previous result is still true, but one needs to consider the isomorphism induced by  $p$  between the substructure of  $\mathfrak{M}$  generated by  $a_1, \dots, a_n$  and the substructure of  $\mathfrak{N}$  generated by  $b_1, \dots, b_n$ .

The previous result can be “broken down” into stages by establishing a connection between the number of nested quantifiers in a formula and how many times the relevant partial isomorphisms can be extended.

**Definition bas.5.** For any formula  $\varphi$ , the *quantifier rank* of  $\varphi$ , denoted by  $\text{qr}(\varphi) \in \mathbb{N}$ , is recursively defined as the highest number of nested quantifiers in  $\varphi$ . Two structures  $\mathfrak{M}$  and  $\mathfrak{N}$  are *n-equivalent*, written  $\mathfrak{M} \equiv_n \mathfrak{N}$ , if they agree on all sentences of quantifier rank less than or equal to  $n$ .

**Proposition bas.6.** Let  $\mathcal{L}$  be a finite purely relational language, i.e., a language containing finitely many predicate symbols and constant symbols, and no function symbols. Then for each  $n \in \mathbb{N}$  there are only finitely many first-order sentences in the language  $\mathcal{L}$  that have quantifier rank no greater than  $n$ , up to logical equivalence.

*mod:bas:pis:  
prop:qr-finite*

*Proof.* By induction on  $n$ .  $\square$

**Definition bas.7.** Given a structure  $\mathfrak{M}$ , let  $|\mathfrak{M}|^{<\omega}$  be the set of all finite sequences over  $|\mathfrak{M}|$ . We use  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$  to range over finite sequences of elements. If  $\mathbf{a} \in |\mathfrak{M}|^{<\omega}$  and  $a \in |\mathfrak{M}|$ , then  $\mathbf{a}a$  represents the *concatenation* of  $\mathbf{a}$  with  $a$ .

**Definition bas.8.** Given structures  $\mathfrak{M}$  and  $\mathfrak{N}$ , we define relations  $I_n \subseteq |\mathfrak{M}|^{<\omega} \times |\mathfrak{N}|^{<\omega}$  between sequences of equal length, by recursion on  $n$  as follows:

1.  $I_0(\mathbf{a}, \mathbf{b})$  if and only if  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the same atomic formulas in  $\mathfrak{M}$  and  $\mathfrak{N}$ ; i.e., if  $s_1(x_i) = a_i$  and  $s_2(x_i) = b_i$  and  $\varphi$  is atomic with all variables among  $x_1, \dots, x_n$ , then  $\mathfrak{M}, s_1 \models \varphi$  if and only if  $\mathfrak{N}, s_2 \models \varphi$ .
2.  $I_{n+1}(\mathbf{a}, \mathbf{b})$  if and only if for every  $a \in A$  there is a  $b \in B$  such that  $I_n(\mathbf{a}a, \mathbf{b}b)$ , and vice-versa.

**Definition bas.9.** Write  $\mathfrak{M} \approx_n \mathfrak{N}$  if  $I_n(A, A)$  holds of  $\mathfrak{M}$  and  $\mathfrak{N}$  (where  $A$  is the empty sequence).

*mod:bas:pis:*  
*thm:b-n-f*

**Theorem bas.10.** *Let  $\mathcal{L}$  be a purely relational language. Then  $I_n(\mathbf{a}, \mathbf{b})$  implies that for every  $\varphi$  such that  $\text{qr}(\varphi) \leq n$ , we have  $\mathfrak{M}, \mathbf{a} \models \varphi$  if and only if  $\mathfrak{N}, \mathbf{b} \models \varphi$  (where again  $\mathbf{a}$  satisfies  $\varphi$  if any  $s$  such that  $s(x_i) = a_i$  satisfies  $\varphi$ ). Moreover, if  $\mathcal{L}$  is finite, the converse also holds.*

*Proof.* The proof that  $I_n(\mathbf{a}, \mathbf{b})$  implies that  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the same formulas of quantifier rank no greater than  $n$  is by an easy induction on  $\varphi$ . For the converse we proceed by induction on  $n$ , using Proposition bas.6, which ensures that for each  $n$  there are at most finitely many non-equivalent formulas of that quantifier rank.

For  $n = 0$  the hypothesis that  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the same quantifier-free formulas gives that they satisfy the same atomic ones, so that  $I_0(\mathbf{a}, \mathbf{b})$ .

For the  $n + 1$  case, suppose that  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the same formulas of quantifier rank no greater than  $n + 1$ ; in order to show that  $I_{n+1}(\mathbf{a}, \mathbf{b})$  suffices to show that for each  $a \in |\mathfrak{M}|$  there is a  $b \in |\mathfrak{N}|$  such that  $I_n(\mathbf{a}a, \mathbf{b}b)$ , and by the inductive hypothesis again suffices to show that for each  $a \in |\mathfrak{M}|$  there is a  $b \in |\mathfrak{N}|$  such that  $\mathbf{a}a$  and  $\mathbf{b}b$  satisfy the same formulas of quantifier rank no greater than  $n$ .

Given  $a \in |\mathfrak{M}|$ , let  $\tau_n^a$  be set of formulas  $\psi(x, \mathbf{y})$  of rank no greater than  $n$  satisfied by  $\mathbf{a}a$  in  $\mathfrak{M}$ ;  $\tau_n^a$  is finite, so we can assume it is a single first-order formula. It follows that  $\mathbf{a}$  satisfies  $\exists x \tau_n^a(x, \mathbf{y})$ , which has quantifier rank no greater than  $n + 1$ . By hypothesis  $\mathbf{b}$  satisfies the same formula in  $\mathfrak{N}$ , so that there is a  $b \in |\mathfrak{N}|$  such that  $\mathbf{b}b$  satisfies  $\tau_n^a$ ; in particular,  $\mathbf{b}b$  satisfies the same formulas of quantifier rank no greater than  $n$  as  $\mathbf{a}a$ . Similarly one shows that for every  $b \in |\mathfrak{N}|$  there is  $a \in |\mathfrak{M}|$  such that  $\mathbf{a}a$  and  $\mathbf{b}b$  satisfy the same formulas of quantifier rank no greater than  $n$ , which completes the proof.  $\square$

*mod:bas:pis:*  
*cor:b-n-f*

**Corollary bas.11.** *If  $\mathfrak{M}$  and  $\mathfrak{N}$  are purely relational structures in a finite language, then  $\mathfrak{M} \approx_n \mathfrak{N}$  if and only if  $\mathfrak{M} \equiv_n \mathfrak{N}$ . In particular  $\mathfrak{M} \equiv \mathfrak{N}$  if and only if for each  $n$ ,  $\mathfrak{M} \approx_n \mathfrak{N}$ .*

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## Bibliography