Theorem bas.1. If a set $\Gamma$ of sentences has arbitrarily large finite models, then it has an infinite model.

Proof. Expand the language of $\Gamma$ by adding countably many new constants $c_0, c_1, \ldots$ and consider the set $\Gamma \cup \{c_i \neq c_j : i \neq j \}$. To say that $\Gamma$ has arbitrarily large finite models means that for every $m > 0$ there is $n \geq m$ such that $\Gamma$ has a model of cardinality $n$. This implies that $\Gamma \cup \{c_i \neq c_j : i \neq j \}$ is finitely satisfiable. By compactness, $\Gamma \cup \{c_i \neq c_j : i \neq j \}$ has a model $M$ whose domain must be infinite, since it satisfies all inequalities $c_i \neq c_j$.

Proposition bas.2. There is no sentence $\phi$ of any first-order language that is true in a structure $M$ if and only if the domain $|M|$ of the structure is infinite.

Proof. If there were such a $\phi$, its negation $\neg\phi$ would be true in all and only the finite structures, and it would therefore have arbitrarily large finite models but it would lack an infinite model, contradicting Theorem bas.1.