bas.1 Non-standard Models of Arithmetic

Definition bas.1. Let $\mathcal{L}_N$ be the language of arithmetic, comprising a constant symbol $\mathfrak{0}$, a 2-place predicate symbol $<$, a 1-place function symbol $'$, and 2-place function symbols $+$ and $\times$.

1. The standard model of arithmetic is the structure $\mathfrak{N}$ for $\mathcal{L}_N$ having $\mathbb{N} = \{0, 1, 2, \ldots\}$ and interpreting $\mathfrak{0}$ as 0, $<$ as the less-than relation over $\mathbb{N}$, and $'$, $+$ and $\times$ as successor, addition, and multiplication over $\mathbb{N}$, respectively.

2. True arithmetic is the theory $\text{Th}(\mathfrak{N})$.

When working in $\mathcal{L}_N$ we abbreviate each term of the form $\mathfrak{0}'\cdots'$, with $n$ applications of the successor function to $\mathfrak{0}$, as $\pi$.

Definition bas.2. A structure $\mathfrak{M}$ for $\mathcal{L}_N$ is standard if and only $\mathfrak{N} \cong \mathfrak{M}$.

Theorem bas.3. There are non-standard enumerable models of true arithmetic.

Proof. Expand $\mathcal{L}_N$ by introducing a new constant symbol $c$, and consider the theory $\text{Th}(\mathfrak{N}) \cup \{ \pi < c : n \in \mathbb{N} \}$.

The theory is finitely satisfiable, so by compactness it has a model $\mathfrak{M}$, which can be taken to be enumerable by the Downward Löwenheim-Skolem theorem. Where $|\mathfrak{M}|$ is the domain of $\mathfrak{M}$, let $\mathfrak{M}$ interpret the non-logical constants of $\mathcal{L}$ as $z = c^{\mathfrak{M}} \in |\mathfrak{M}|$, $\prec = \mathfrak{0}^{\mathfrak{M}} \subseteq |\mathfrak{M}|$, $\ast = \mathfrak{r}^{\mathfrak{M}} : |\mathfrak{M}| \to |\mathfrak{M}|$, and $\oplus = +^{\mathfrak{M}}$, $\otimes = \times^{\mathfrak{M}} : |\mathfrak{M}|^2 \to |\mathfrak{M}|$. For each $x \in |\mathfrak{M}|$, we write $x^*$ for the element of $|\mathfrak{M}|$ obtained from $x$ by application of $\ast$.

Now, if $h$ were an isomorphism of $\mathfrak{N}$ and $\mathfrak{M}$, there would be $n \in \mathbb{N}$ such that $h(n) = c^{\mathfrak{M}}$. So let $s$ be any assignment in $\mathfrak{N}$ such that $s(x) = n$. Then $\mathfrak{N}, s \models \pi = x$; by the proof of ??, also $\mathfrak{M}, h \circ s \models \pi = x$, so that $c^{\mathfrak{M}} = z^*\cdots^*$ (with $\ast$ iterated $n$ times). But this is impossible since by assumption $\mathfrak{M} \models \pi < c$ and $\prec$ is irreflexive. So $\mathfrak{M}$ is non-standard. \square

Problem bas.1. A relation $R$ over a set $X$ is well-founded if and only if there are no infinite descending chains in $R$, i.e., if there are no $x_0, x_1, x_2, \ldots$ in $X$ such that $x_nRx_{n+1}$ for all $n \in \mathbb{N}$. Assuming Zermelo-Fraenkel set theory $ZF$ is consistent, show that there are non-well-founded models of $ZF$, i.e., models $\mathfrak{M}$ such that $\ldots x_2Rx_1Rx_0$. Since the non-standard model $\mathfrak{M}$ from Theorem bas.3 is elementarily equivalent to the standard one, a number of properties of $\mathfrak{M}$ can be derived. The rest of this section is devoted to such a task, which will allow us to obtain a precise characterization of enumerable non-standard models of $\text{Th}(\mathfrak{N})$.

1. No member of $|\mathfrak{M}|$ is $\prec$-less than itself: the sentence $\forall x \neg x < x$ is true in $\mathfrak{N}$ and therefore in $\mathfrak{M}$.
2. By a similar reasoning we obtain that $\prec$ is a \textit{linear ordering} of $|\mathcal{M}|$, i.e., a total, irreflexive, transitive relation on $|\mathcal{M}|$.

3. The element $z$ is the $\prec$-least element of $|\mathcal{M}|$.

4. Any member of $|\mathcal{M}|$ is $\prec$-less than its $\ast$-successor and $x^\ast$ is the $\prec$-least member of $|\mathcal{M}|$ greater than $x$.

5. $\mathcal{M}$ contains an initial segment (of $\prec$) isomorphic to $\mathbb{N}$: $z, z^\ast, z^{\ast\ast}, \ldots$, which we call the \textit{standard part} of $|\mathcal{M}|$. Any other member of $|\mathcal{M}|$ is \textit{non-standard}. There must be non-standard members of $|\mathcal{M}|$, or else the function $h$ from the proof of Theorem bas.3 is an isomorphism. We use $n, m, \ldots$ as variables ranging on this standard part of $\mathcal{M}$.

6. Every non-standard element is greater than any standard one; this is because for every $n \in \mathbb{N},$

\[
\mathcal{N} \models \forall z (\neg (z = 0 \lor \cdots \lor z = \pi) \rightarrow \pi < z),
\]

so if $z \in |\mathcal{M}|$ is different from all the standard elements, it must be \textit{greater} than all of them.

7. Any member of $|\mathcal{M}|$ other than $z$ is the $\ast$-successor of some unique element of $|\mathcal{M}|$, denoted by $^*x$. If $x = y^*$ then both $x$ and $y$ are standard if one of them is (and both non-standard if one of them is).

8. Define an equivalence relation $\equiv$ over $|\mathcal{M}|$ by saying that $x \equiv y$ if and only if for some standard $n$, either $x \oplus n = y$ or $y \oplus n = x$. In other words, $x \equiv y$ if and only if $x$ and $y$ are a finite distance apart. If $n$ and $m$ are standard then $n \equiv m$. Define the \textit{block} of $x$ to be the equivalence class $[x] = \{ y : x \equiv y \}$.

9. Suppose that $x \prec y$ where $x \neq y$. Since $\mathcal{N} \models \forall x \forall y (x < y \rightarrow (x' < y \lor x' = y))$, either $x^\ast \prec y$ or $x^\ast = y$. The latter is impossible because it implies $x \equiv y$, so $x \prec y$. Similarly, if $x \prec y$ and $x \neq y$, then $x \prec y^\ast$. Therefore if $x \prec y$ and $x \neq y$, then every $w \approx x$ is $\prec$-less than every $v \approx y$. Accordingly, each block $[x]$ forms a doubly infinite chain

\[
\cdots \prec \ast \ast x \prec \ast x \prec x^\ast \prec x^{\ast\ast} \prec \cdots
\]

which is referred to as a Z-chain because it has the order type of the integers.

10. The $\prec$ ordering can be lifted up to the blocks: if $x \prec y$ then the block of $x$ is less than the block of $y$. A block is \textit{non-standard} if it contains a non-standard element. The standard block is the least block.

11. There is no least non-standard block: if $y$ is non-standard then there is a $x \prec y$ where $x$ is also non-standard and $x \neq y$. Proof: in the standard model $\mathcal{N}$, every number is divisible by two, possibly with remainder one,
i.e., $\mathfrak{M} \models \forall y \forall x \ (y = x + x \lor y = x + x + 0')$. By elementary equivalence, for every $y \in |\mathfrak{M}|$ there is $x \in |\mathfrak{M}|$ such that either $x \oplus x = y$ or $x \oplus x \oplus z^* = y$. If $x$ were standard, then so would be $y$; so $x$ is non-standard. Furthermore, $x$ and $y$ belong to different blocks, i.e., $x \not\approx y$. To see this, assume they did belong to the same block, i.e., $x \oplus n = y$ for some standard $n$. If $y = x \oplus x$, then $x \oplus n = x \oplus x$, whence $x = n$ by the cancellation law for addition (which holds in $\mathbb{N}$ and therefore in $\mathfrak{M}$ as well), and $x$ would be standard after all. Similarly if $y = x \oplus x \oplus z^*$.

12. By a similar argument, there is no greatest block.

13. The ordering of the blocks is dense: if $[x]$ is less than $[y]$ (where $x \not\approx y$), then there is a block $[z]$ distinct from both that is between them. Suppose $x \prec y$. As before, $x \oplus y$ is divisible by two (possibly with remainder) so there is a $u \in |\mathfrak{M}|$ such that either $x \oplus y = u \oplus u$ or $x \oplus y = u \oplus u \oplus z^*$. The element $u$ is the average of $x$ and $y$, and so is between them. Assume $x \oplus y = u \oplus u$ (the other case being similar): if $u \approx x$ then for some standard $n$:

$$x \oplus y = x \oplus n \oplus x \oplus n,$$

so $y = x \oplus n \oplus n$ and we would have $x \approx y$, against assumption. We conclude that $u \not\approx x$. A similar argument gives $u \not\approx y$.

The non-standard blocks are therefore ordered like the rationals: they form an enumerable linear ordering without endpoints. It follows that for any two enumerable non-standard models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of true arithmetic, their reducts to the language containing $<$ and $=$ only are isomorphic. Indeed, an isomorphism $h$ can be defined as follows: the standard parts of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic to the standard model $\mathbb{N}$ and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore by $\approx$ are isomorphic; an isomorphism of the blocks can be extended to an isomorphism within the blocks by matching up arbitrary elements in each, and then taking the image of the successor of $x$ in $\mathfrak{M}_1$ to be the successor of the image of $x$ in $\mathfrak{M}_2$. Note that it does not follow that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic in the full language of arithmetic (indeed, isomorphism is always relative to a signature), as there are non-isomorphic ways to define addition and multiplication over $|\mathfrak{M}_1|$ and $|\mathfrak{M}_2|$. (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be $2$.)

**Problem bas.2.** Show that there can be no greatest block in a non-standard model of arithmetic.

**Problem bas.3.** Let $\mathcal{L}$ be the first-order language containing $<$ as its only predicate symbol (besides $=$), and let $\mathfrak{N} = (\mathbb{N}, <)$. All the finite or cofinite subsets of $\mathfrak{N}$ are definable. Show that these are the only definable subsets of $\mathfrak{N}$. 

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(Hint: First, let $\text{prc}(x,y)$ be the $L$-formula abbreviating “$x$ is the immediate predecessor of $y$”:

$$x < y \land \neg \exists z (x < z \land z < y).$$

Now, to any definable subset of $\mathcal{N}$ there corresponds a formula $\varphi(x)$ in $L$. For any such $\varphi$, consider the sentence $\theta$:

$$\exists x \forall y \forall z ((x < y \land x < z \land \text{prc}(y,z) \land \varphi(y)) \rightarrow \varphi(z)).$$

Show that $\mathcal{N} \models \theta$ if and only if the subset of $\mathcal{N}$ defined by $\varphi$ is either finite or cofinite.

Now, let $\mathcal{M}$ be a non-standard model elementarily equivalent to $\mathcal{N}$. If $a \in |\mathcal{M}|$ is non-standard, let $b,c \in |\mathcal{M}|$ be greater than $a$, and let $b$ be the immediate predecessor of $c$. Then there is an automorphism $h$ of $|\mathcal{M}|$ such that $h(b) = c$ (why?). Therefore, if $b$ satisfies $\varphi$, so does $c$ (why?). It follows that $\theta$ is true in $\mathcal{M}$, and hence also in $\mathcal{N}$. But this implies that the subset of $\mathcal{N}$ defined by $\varphi$ is either finite or co-finite.

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**Bibliography**