

## bas.1 Isomorphic Structures

mod:bas:iso:  
sec

First-order **structures** can be alike in one of two ways. One way in which they can be alike is that they make the same **sentences** true. We call such **structures** *elementarily equivalent*. But structures can be very different and still make the same **sentences** true—for instance, one can be **enumerable** and the other not. This is because there are lots of features of a **structure** that cannot be expressed in first-order languages, either because the language is not rich enough, or because of fundamental limitations of first-order logic such as the Löwenheim-Skolem theorem. So another, stricter, aspect in which **structures** can be alike is if they are fundamentally the same, in the sense that they only differ in the objects that make them up, but not in their structural features. A way of making this precise is by the notion of an *isomorphism*.

mod:bas:iso:  
defn:elem-equiv

**Definition bas.1.** Given two **structures**  $\mathfrak{M}$  and  $\mathfrak{M}'$  for the same **language**  $\mathcal{L}$ , we say that  $\mathfrak{M}$  is *elementarily equivalent* to  $\mathfrak{M}'$ , written  $\mathfrak{M} \equiv \mathfrak{M}'$ , if and only if for every **sentence**  $\varphi$  of  $\mathcal{L}$ ,  $\mathfrak{M} \models \varphi$  iff  $\mathfrak{M}' \models \varphi$ .

mod:bas:iso:  
defn:isomorphism

**Definition bas.2.** Given two **structures**  $\mathfrak{M}$  and  $\mathfrak{M}'$  for the same **language**  $\mathcal{L}$ , we say that  $\mathfrak{M}$  is *isomorphic* to  $\mathfrak{M}'$ , written  $\mathfrak{M} \simeq \mathfrak{M}'$ , if and only if there is a function  $h: |\mathfrak{M}| \rightarrow |\mathfrak{M}'|$  such that:

1.  $h$  is **injective**: if  $h(x) = h(y)$  then  $x = y$ ;
2.  $h$  is **surjective**: for every  $y \in |\mathfrak{M}'|$  there is  $x \in |\mathfrak{M}|$  such that  $h(x) = y$ ;
3. for every **constant symbol**  $c$ :  $h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ ;
4. for every  $n$ -place **predicate symbol**  $P$ :

mod:bas:iso:  
defn:iso-const  
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defn:iso-pred

$$\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}} \quad \text{iff} \quad \langle h(a_1), \dots, h(a_n) \rangle \in P^{\mathfrak{M}'};$$

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defn:iso-func

5. for every  $n$ -place **function symbol**  $f$ :

$$h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{M}'}(h(a_1), \dots, h(a_n)).$$

mod:bas:iso:  
thm:isom

**Theorem bas.3.** *If  $\mathfrak{M} \simeq \mathfrak{M}'$  then  $\mathfrak{M} \equiv \mathfrak{M}'$ .*

*Proof.* Let  $h$  be an isomorphism of  $\mathfrak{M}$  onto  $\mathfrak{M}'$ . For any assignment  $s$ ,  $h \circ s$  is the composition of  $h$  and  $s$ , i.e., the assignment in  $\mathfrak{M}'$  such that  $(h \circ s)(x) = h(s(x))$ . By induction on  $t$  and  $\varphi$  one can prove the stronger claims:

- a.  $h(\text{Val}_s^{\mathfrak{M}}(t)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(t)$ .
- b.  $\mathfrak{M}, s \models \varphi$  iff  $\mathfrak{M}', h \circ s \models \varphi$ .

The first is proved by induction on the complexity of  $t$ .

1. If  $t \equiv c$ , then  $\text{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}}$  and  $\text{Val}_{h \circ s}^{\mathfrak{M}'}(c) = c^{\mathfrak{M}'}$ . Thus,  $h(\text{Val}_s^{\mathfrak{M}}(t)) = h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$  (by (3) of **Definition bas.2**) =  $\text{Val}_{h \circ s}^{\mathfrak{M}'}(t)$ .

2. If  $t \equiv x$ , then  $\text{Val}_s^{\mathfrak{M}}(x) = s(x)$  and  $\text{Val}_{h \circ s}^{\mathfrak{M}'}(x) = h(s(x))$ . Thus,  $h(\text{Val}_s^{\mathfrak{M}}(x)) = h(s(x)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(x)$ .
3. If  $t \equiv f(t_1, \dots, t_n)$ , then

$$\begin{aligned} \text{Val}_s^{\mathfrak{M}}(t) &= f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)) \quad \text{and} \\ \text{Val}_{h \circ s}^{\mathfrak{M}'}(t) &= f^{\mathfrak{M}'}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)). \end{aligned}$$

The induction hypothesis is that for each  $i$ ,  $h(\text{Val}_s^{\mathfrak{M}}(t_i)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_i)$ . So,

$$\begin{aligned} h(\text{Val}_s^{\mathfrak{M}}(t)) &= h(f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n))) \\ &= h(f^{\mathfrak{M}}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n))) && (1) \quad \text{mod:bas:iso:} \\ &= f^{\mathfrak{M}'}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)) && (2) \quad \text{iso-1} \\ &= \text{Val}_{h \circ s}^{\mathfrak{M}'}(t) && \text{iso-2} \end{aligned}$$

Here, eq. (1) follows by induction hypothesis and eq. (2) by (5) of Definition bas.2.

Part (2) is left as an exercise.

If  $\varphi$  is a sentence, the assignments  $s$  and  $h \circ s$  are irrelevant, and we have  $\mathfrak{M} \models \varphi$  iff  $\mathfrak{M}' \models \varphi$ .  $\square$

**Problem bas.1.** Carry out the proof of (b) of Theorem bas.3 in detail. Make sure to note where each of the five properties characterizing isomorphisms of Definition bas.2 is used.

**Definition bas.4.** An *automorphism* of a structure  $\mathfrak{M}$  is an isomorphism of  $\mathfrak{M}$  onto itself.

**Problem bas.2.** Show that for any structure  $\mathfrak{M}$ , if  $X$  is a definable subset of  $\mathfrak{M}$ , and  $h$  is an automorphism of  $\mathfrak{M}$ , then  $X = \{h(x) : x \in X\}$  (i.e.,  $X$  is fixed under  $h$ ).

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## Bibliography