## bas.1 Isomorphic Structures

mod:bas:iso: sec

First-order structures can be alike in one of two ways. One way in which the can be alike is that they make the same sentences true. We call such structures *elementarily equivalent*. But structures can be very different and still make the same sentences true—for instance, one can be enumerable and the other not. This is because there are lots of features of a structure that cannot be expressed in first-order languages, either because the language is not rich enough, or because of fundamental limitations of first-order logic such as the Löwenheim-Skolem theorem. So another, stricter, aspect in which structures can be alike is if they are fundamentally the same, in the sense that they only differ in the objects that make them up, but not in their structural features. A way of making this precise is by the notion of an *isomorphism*.

mod:bas:iso: defn:elem-equiv

**Definition bas.1.** Given two structures $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is *elementarily equivalent to* $\mathfrak{M}'$, written $\mathfrak{M} \equiv \mathfrak{M}'$, if and only if for every sentence $\varphi$ of $\mathcal{L}$, $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$.

mod:bas:iso: defn:isomorphism

**Definition bas.2.** Given two structures $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is *isomorphic to* $\mathfrak{M}'$, written $\mathfrak{M} \simeq \mathfrak{M}'$, if and only if there is a function $h\colon |\mathfrak{M}| \to |\mathfrak{M}'|$ such that:

1. $h$ is injective: if $h(x) = h(y)$ then $x = y$;
2. $h$ is surjective: for every $y \in |\mathfrak{M}'|$ there is $x \in |\mathfrak{M}|$ such that $h(x) = y$;
3. mod:bas:iso: defn:iso-const for every constant symbol $c$: $h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$;
4. mod:bas:iso: defn:iso-pred for every $n$-place predicate symbol $P$:
$$\langle a_1, \dots, a_n\rangle \in P^{\mathfrak{M}} \quad \text{iff} \quad \langle h(a_1), \dots, h(a_n)\rangle \in P^{\mathfrak{M}'};$$
5. mod:bas:iso: defn:iso-func for every $n$-place function symbol $f$:
$$h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{M}'}(h(a_1), \dots, h(a_n)).$$

*mod:bas:iso: thm:isom*

**Theorem bas.3.** *If* $\mathfrak{M} \simeq \mathfrak{M}'$ *then* $\mathfrak{M} \equiv \mathfrak{M}'$.

*Proof.* Let $h$ be an isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$. For any assignment $s$, $h \circ s$ is the composition of $h$ and $s$, i.e., the assignment in $\mathfrak{M}'$ such that $(h \circ s)(x) = h(s(x))$. By induction on $t$ and $\varphi$ one can prove the stronger claims:

a. $h(\mathrm{Val}_s^{\mathfrak{M}}(t)) = \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t)$.

b. $\mathfrak{M}, s \vDash \varphi$ iff $\mathfrak{M}', h \circ s \vDash \varphi$.

The first is proved by induction on the complexity of $t$.

1. If $t \equiv c$, then $\mathrm{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}}$ and $\mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(c) = c^{\mathfrak{M}'}$. Thus, $h(\mathrm{Val}_s^{\mathfrak{M}}(t)) = h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ (by (3) of Definition bas.2) $= \mathrm{Val}_{h\circ s}^{\mathfrak{M}'}(t)$.

2. If $t \equiv x$, then $\mathrm{Val}^{\mathfrak{M}}_{s}(x) = s(x)$ and $\mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(x) = h(s(x))$. Thus, $h(\mathrm{Val}^{\mathfrak{M}}_{s}(x)) = h(s(x)) = \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(x)$.

3. If $t \equiv f(t_1, \dots, t_n)$, then

$$\begin{aligned} \mathrm{Val}^{\mathfrak{M}}_{s}(t) &= f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}}_{s}(t_n)) \quad \text{and} \\ \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t) &= f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t_n)). \end{aligned}$$

   The induction hypothesis is that for each $i$, $h(\mathrm{Val}^{\mathfrak{M}}_{s}(t_i)) = \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t_i)$. So,

$$\begin{aligned} h(\mathrm{Val}^{\mathfrak{M}}_{s}(t)) &= h(f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}}_{s}(t_n)) \\ &= h(f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t_n)) && (1) \\ &= f^{\mathfrak{M}'}(\mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t_n)) && (2) \\ &= \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t) \end{aligned}$$

mod:bas:iso:iso-1

mod:bas:iso:iso-2

   Here, eq. (1) follows by induction hypothesis and eq. (2) by (5) of Definition bas.2.

Part (b) is left as an exercise.

If $\varphi$ is a sentence, the assignments $s$ and $h \circ s$ are irrelevant, and we have $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$. □

**Problem bas.1.** Carry out the proof of (b) of Theorem bas.3 in detail. Make sure to note where each of the five properties characterizing isomorphisms of Definition bas.2 is used.

**Definition bas.4.** An *automorphism* of a structure $\mathfrak{M}$ is an isomorphism of $\mathfrak{M}$ onto itself.

**Problem bas.2.** Show that for any structure $\mathfrak{M}$, if $X$ is a definable subset of $\mathfrak{M}$, and $h$ is an automorphism of $\mathfrak{M}$, then $X = \{h(x) : x \in X\}$ (i.e., $X$ is fixed under $h$).

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## Bibliography