**bas.1 Dense Linear Orders**

**Definition bas.1.** A dense linear ordering without endpoints is a structure $\mathcal{M}$ for the language containing a single 2-place predicate symbol $<$ satisfying the following sentences:

1. $\forall x \neg x < x$;
2. $\forall x \forall y \forall z (x < y \rightarrow (y < z \rightarrow x < z))$;
3. $\forall x \forall y (x < y \lor x = y \lor y < x)$;
4. $\forall x \exists y x < y$;
5. $\forall x \exists y y < x$;
6. $\forall x \forall y (x < y \rightarrow \exists z (x < z \land z < y))$.

**Theorem bas.2.** Any two enumerable dense linear orderings without endpoints are isomorphic.

**Proof.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be enumerable dense linear orderings without endpoints, with $<_1 = <^{\mathcal{M}_1}$ and $<_2 = <^{\mathcal{M}_2}$, and let $I$ be the set of all partial isomorphisms between them. $I$ is not empty since at least $\emptyset \in I$. We show that $I$ satisfies the Back-and-Forth property. Then $\mathcal{M}_1 \simeq_p \mathcal{M}_2$, and the theorem follows by $??$.

To show $I$ satisfies the Forth property, let $p \in I$ and let $p(a_i) = b_i$ for $i = 1, \ldots, n$, and without loss of generality suppose $a_1 <_1 a_2 <_1 \cdots <_1 a_n$. Given $a \in |\mathcal{M}_1|$, find $b \in |\mathcal{M}_2|$ as follows:

1. if $a <_2 a_1$ let $b \in |\mathcal{M}_2|$ be such that $b <_2 b_1$;
2. if $a_n <_1 a$ let $b \in |\mathcal{M}_2|$ be such that $b_n <_2 b$;
3. if $a_i <_1 a <_1 a_{i+1}$ for some $i$, then let $b \in |\mathcal{M}_2|$ be such that $b_i <_2 b <_2 b_{i+1}$.

It is always possible to find a $b$ with the desired property since $\mathcal{M}_2$ is a dense linear ordering without endpoints. Define $q = p \cup \{(a, b)\}$ so that $q \in I$ is the desired extension of $p$. This establishes the Forth property. The Back property is similar. So $\mathcal{M}_1 \simeq_p \mathcal{M}_2$; by $??, \mathcal{M}_1 \simeq \mathcal{M}_2$.

**Problem bas.1.** Complete the proof of Theorem bas.2 by verifying that $I$ satisfies the Back property.

**Remark 1.** Let $\mathcal{G}$ be any enumerable dense linear ordering without endpoints. Then (by Theorem bas.2) $\mathcal{G} \simeq \mathcal{Q}$, where $\mathcal{Q} = (\mathbb{Q}, <)$ is the enumerable dense linear ordering having the set $\mathbb{Q}$ of the rational numbers as its domain. Now consider again the structure $\mathcal{R} = (\mathbb{R}, <)$ from $??$. We saw that there is an enumerable structure $\mathcal{G}$ such that $\mathcal{R} \equiv \mathcal{G}$. But $\mathcal{G}$ is an enumerable dense linear
ordering without endpoints, and so it is isomorphic (and hence elementarily equivalent) to the structure $\Omega$. By transitivity of elementary equivalence, $R \equiv \Omega$. (We could have shown this directly by establishing $R \simeq_p \Omega$ by the same back-and-forth argument.)

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Bibliography