

## bas.1 Dense Linear Orders

**Definition bas.1.** A *dense linear ordering without endpoints* is a structure  $\mathfrak{M}$  for the language containing a single 2-place predicate symbol  $<$  satisfying the following sentences:

1.  $\forall x x < x$ ;
2.  $\forall x \forall y \forall z (x < y \rightarrow (y < z \rightarrow x < z))$ ;
3.  $\forall x \forall y (x < y \vee x = y \vee y < x)$ ;
4.  $\forall x \exists y x < y$ ;
5.  $\forall x \exists y y < x$ ;
6.  $\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y))$ .

*mod:bas:dlo:* **Theorem bas.2.** Any two *enumerable* dense linear orderings without endpoints are isomorphic.  
*thm:cantorQ*

*Proof.* Let  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be *enumerable* dense linear orderings without endpoints, with  $<_1 = <^{\mathfrak{M}_1}$  and  $<_2 = <^{\mathfrak{M}_2}$ , and let  $\mathcal{I}$  be the set of all partial isomorphisms between them.  $\mathcal{I}$  is not empty since at least  $\emptyset \in \mathcal{I}$ . We show that  $\mathcal{I}$  satisfies the Back-and-Forth property. Then  $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$ , and the theorem follows by ??.

To show  $\mathcal{I}$  satisfies the Forth property, let  $p \in \mathcal{I}$  and let  $p(a_i) = b_i$  for  $i = 1, \dots, n$ , and without loss of generality suppose  $a_1 <_1 a_2 <_1 \dots <_1 a_n$ . Given  $a \in |\mathfrak{M}_1|$ , find  $b \in |\mathfrak{M}_2|$  as follows:

1. if  $a <_2 a_1$  let  $b \in |\mathfrak{M}_2|$  be such that  $b <_2 b_1$ ;
2. if  $a_n <_1 a$  let  $b \in |\mathfrak{M}_2|$  be such that  $b_n <_2 b$ ;
3. if  $a_i <_1 a <_1 a_{i+1}$  for some  $i$ , then let  $b \in |\mathfrak{M}_2|$  be such that  $b_i <_2 b <_2 b_{i+1}$ .

It is always possible to find a  $b$  with the desired property since  $\mathfrak{M}_2$  is a dense linear ordering without endpoints. Define  $q = p \cup \{(a, b)\}$  so that  $q \in \mathcal{I}$  is the desired extension of  $p$ . This establishes the Forth property. The Back property is similar. So  $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$ ; by ??,  $\mathfrak{M}_1 \simeq \mathfrak{M}_2$ .  $\square$

**Problem bas.1.** Complete the proof of [Theorem bas.2](#) by verifying that  $\mathcal{I}$  satisfies the Back property.

*Remark 1.* Let  $\mathfrak{S}$  be any *enumerable* dense linear ordering without endpoints. Then (by [Theorem bas.2](#))  $\mathfrak{S} \simeq \mathfrak{Q}$ , where  $\mathfrak{Q} = (\mathbb{Q}, <)$  is the *enumerable* dense linear ordering having the set  $\mathbb{Q}$  of the rational numbers as its domain. Now consider again the structure  $\mathfrak{R} = (\mathbb{R}, <)$  from ???. We saw that there is an *enumerable* structure  $\mathfrak{S}$  such that  $\mathfrak{R} \equiv \mathfrak{S}$ . But  $\mathfrak{S}$  is an *enumerable* dense linear

ordering without endpoints, and so it is isomorphic (and hence elementarily equivalent) to the **structure**  $\mathfrak{Q}$ . By transitivity of elementary equivalence,  $\mathfrak{R} \equiv \mathfrak{Q}$ . (We could have shown this directly by establishing  $\mathfrak{R} \simeq_p \mathfrak{Q}$  by the same back-and-forth argument.)

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## Bibliography