bas.1 Dense Linear Orders

Definition bas.1. A dense linear ordering without endpoints is a structure \( M \) for the language containing a single 2-place predicate symbol \( < \) satisfying the following sentences:

1. \( \forall x \ x < x \);
2. \( \forall x \ \forall y \ \forall z (x < y \rightarrow (y < z \rightarrow x < z)) \);
3. \( \forall x \ \forall y (x < y \lor x = y \lor y < x) \);
4. \( \forall x \ \exists y x < y \);
5. \( \forall x \ \exists y y < x \);
6. \( \forall x \ \forall y (x < y \rightarrow \exists z (x < z \land z < y)) \).

Theorem bas.2. Any two enumerable dense linear orderings without endpoints are isomorphic.

Proof. Let \( M_1 \) and \( M_2 \) be enumerable dense linear orderings without endpoints, with \( <_1 = <_{M_1} \) and \( <_2 = <_{M_2} \), and let \( I \) be the set of all partial isomorphisms between them. \( I \) is not empty since at least \( \emptyset \in I \). We show that \( I \) satisfies the Back-and-Forth property. Then \( M_1 \simeq_p M_2 \), and the theorem follows by ??.

To show \( I \) satisfies the Forth property, let \( p \in I \) and let \( p(a_i) = b_i \) for \( i = 1, \ldots, n \), and without loss of generality suppose \( a_1 <_1 a_2 <_1 \cdots <_1 a_n \). Given \( a \in \lvert M_1 \rvert \), find \( b \in \lvert M_2 \rvert \) as follows:

1. if \( a <_2 a_1 \) let \( b \in \lvert M_2 \rvert \) be such that \( b <_2 b_1 \);
2. if \( a_n <_1 a \) let \( b \in \lvert M_2 \rvert \) be such that \( b_n <_2 b \);
3. if \( a_i <_1 a <_1 a_{i+1} \) for some \( i \), then let \( b \in \lvert M_2 \rvert \) be such that \( b_i <_2 b <_2 b_{i+1} \).

It is always possible to find a \( b \) with the desired property since \( M_2 \) is a dense linear ordering without endpoints. Define \( q = p \cup \{(a, b)\} \) so that \( q \in I \) is the desired extension of \( p \). This establishes the Forth property. The Back property is similar. So \( M_1 \simeq_p M_2 \); by ??, \( M_1 \simeq M_2 \).

Problem bas.1. Complete the proof of Theorem bas.2 by verifying that \( I \) satisfies the Back property.

Remark 1. Let \( S \) be any enumerable dense linear ordering without endpoints. Then (by Theorem bas.2) \( S \simeq \mathbb{Q} \), where \( \mathbb{Q} = (\mathbb{Q}, <) \) is the enumerable dense linear ordering having the set \( \mathbb{Q} \) of the rational numbers as its domain. Now consider again the structure \( R = (\mathbb{R}, <) \) from ??.. We saw that there is an enumerable structure \( S \) such that \( R \equiv S \). But \( S \) is an enumerable dense linear
ordering without endpoints, and so it is isomorphic (and hence elementarily equivalent) to the structure $\Omega$. By transitivity of elementary equivalence, $\mathcal{R} \equiv \Omega$. (We could have shown this directly by establishing $\mathcal{R} \simeq_p \Omega$ by the same back-and-forth argument.)

Photo Credits

Bibliography