bas.1 Dense Linear Orders

Definition bas.1. A *dense linear ordering without endpoints* is a structure $\mathfrak{M}$ for the language containg a single 2-place predicate symbol $<$ satisfying the following sentences:

1. $\forall x\; x < x$;
2. $\forall x\, \forall y\, \forall z\, (x < y \to (y < z \to x < z))$;
3. $\forall x\, \forall y\, (x < y \lor x = y \lor y < x)$;
4. $\forall x\, \exists y\, x < y$;
5. $\forall x\, \exists y\, y < x$;
6. $\forall x\, \forall y\, (x < y \to \exists z\, (x < z \land z < y))$.

mod:bas:dlo:thm:cantorQ **Theorem bas.2.** *Any two enumerable dense linear orderings without endpoints are isomorphic.*

Proof. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be enumerable dense linear orderings without endpoints, with $<_1 = <^{\mathfrak{M}_1}$ and $<_2 = <^{\mathfrak{M}_2}$, and let $\mathcal{I}$ be the set of all partial isomorphisms between them. $\mathcal{I}$ is not empty since at least $\emptyset \in \mathcal{I}$. We show that $\mathcal{I}$ satisfies the Back-and-Forth property. Then $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$, and the theorem follows by **??**.

To show $\mathcal{I}$ satisifes the Forth property, let $p \in \mathcal{I}$ and let $p(a_i) = b_i$ for $i = 1$, …, n, and without loss of generality suppose $a_1 <_1 a_2 <_1 \cdots <_1 a_n$. Given $a \in |\mathfrak{M}_1|$, find $b \in |\mathfrak{M}_2|$ as follows:

1. if $a <_2 a_1$ let $b \in |\mathfrak{M}_2|$ be such that $b <_2 b_1$;
2. if $a_n <_1 a$ let $b \in |\mathfrak{M}_2|$ be such that $b_n <_2 b$;
3. if $a_i <_1 a <_1 a_{i+1}$ for some i, then let $b \in |\mathfrak{M}_2|$ be such that $b_i <_2 b <_2 b_{i+1}$.

It is always possible to find a b with the desired property since $\mathfrak{M}_2$ is a dense linear ordering without endpoints. Define $q = p \cup \{\langle a, b\rangle\}$ so that $q \in \mathcal{I}$ is the desired extension of p. This establishes the Forth property. The Back property is similar. So $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$; by **??**, $\mathfrak{M}_1 \simeq \mathfrak{M}_2$. □

Problem bas.1. Complete the proof of Theorem bas.2 by verifying that $\mathcal{I}$ satisfies the Back property.

Remark 1. Let $\mathfrak{S}$ be any enumerable dense linear ordering without endpoints. Then (by Theorem bas.2) $\mathfrak{S} \simeq \mathfrak{Q}$, where $\mathfrak{Q} = (\mathbb{Q}, <)$ is the enumerable dense linear ordering having the set $\mathbb{Q}$ of the rational numbers as its domain. Now consider again the structure $\mathfrak{R} = (\mathbb{R}, <)$ from **??**. We saw that there is an enumerable structure $\mathfrak{S}$ such that $\mathfrak{R} \equiv \mathfrak{S}$. But $\mathfrak{S}$ is an enumerable dense linear ordering without endpoints, and so it is isomorphic (and hence elementarily equivalent) to the structure $\mathfrak{Q}$. By transitivity of elementary equivalence, $\mathfrak{R} \equiv \mathfrak{Q}$. (We could have shown this directly by establishing $\mathfrak{R} \simeq_p \mathfrak{Q}$ by the same back-and-forth argument.)

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Bibliography