

Chapter udf

Basics of Model Theory

bas.1 Reducts and Expansions

Often it is useful or necessary to compare languages which have symbols in common, as well as **structures** for these languages. The most common case is when all the symbols in a language \mathcal{L} are also part of a language \mathcal{L}' , i.e., $\mathcal{L} \subseteq \mathcal{L}'$. An \mathcal{L} -**structure** \mathfrak{M} can then always be expanded to an \mathcal{L}' -**structure** by adding interpretations of the additional symbols while leaving the interpretations of the common symbols the same. On the other hand, from an \mathcal{L}' -structure \mathfrak{M}' we can obtain an \mathcal{L} -structure simply by “forgetting” the interpretations of the symbols that do not occur in \mathcal{L} .

mod:bas:red:
defn:reduct **Definition bas.1.** Suppose $\mathcal{L} \subseteq \mathcal{L}'$, \mathfrak{M} is an \mathcal{L} -**structure** and \mathfrak{M}' is an \mathcal{L}' -**structure**. \mathfrak{M} is the *reduct* of \mathfrak{M}' to \mathcal{L} , and \mathfrak{M}' is an *expansion* of \mathfrak{M} to \mathcal{L}' iff

1. $|\mathfrak{M}| = |\mathfrak{M}'|$
2. For every **constant symbol** $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$.
3. For every **function symbol** $f \in \mathcal{L}$, $f^{\mathfrak{M}} = f^{\mathfrak{M}'}$.
4. For every **predicate symbol** $P \in \mathcal{L}$, $P^{\mathfrak{M}} = P^{\mathfrak{M}'}$.

mod:bas:red:
prop:reduct **Proposition bas.2.** If an \mathcal{L} -**structure** \mathfrak{M} is a reduct of an \mathcal{L}' -**structure** \mathfrak{M}' , then for all \mathcal{L} -**sentences** φ ,

$$\mathfrak{M} \models \varphi \text{ iff } \mathfrak{M}' \models \varphi.$$

Proof. Exercise. □

Problem bas.1. Prove Proposition bas.2.

Definition bas.3. When we have an \mathcal{L} -structure \mathfrak{M} , and $\mathcal{L}' = \mathcal{L} \cup \{P\}$ is the expansion of \mathcal{L} obtained by adding a single n -place **predicate symbol** P , and $R \subseteq |\mathfrak{M}|^n$ is an n -place relation, then we write (\mathfrak{M}, R) for the expansion \mathfrak{M}' of \mathfrak{M} with $P^{\mathfrak{M}'} = R$.

bas.2 Substructures

The **domain** of a **structure** \mathfrak{M} may be a subset of another \mathfrak{M}' . But we should obviously only consider \mathfrak{M} a “part” of \mathfrak{M}' if not only $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$, but \mathfrak{M} and \mathfrak{M}' “agree” in how they interpret the symbols of the language at least on the shared part $|\mathfrak{M}|$. mod:bas:sub:
sec

Definition bas.4. Given **structures** \mathfrak{M} and \mathfrak{M}' for the same language \mathcal{L} , we say that \mathfrak{M} is a *substructure* of \mathfrak{M}' , and \mathfrak{M}' an *extension* of \mathfrak{M} , written $\mathfrak{M} \subseteq \mathfrak{M}'$, iff mod:bas:sub:
defn:substructure

1. $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$,
2. For each constant $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$;
3. For each n -place **predicate symbol** $f \in \mathcal{L}$ $f^{\mathfrak{M}}(a_1, \dots, a_n) = f^{\mathfrak{M}'}(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in |\mathfrak{M}|$.
4. For each n -place **predicate symbol** $R \in \mathcal{L}$, $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{M}}$ iff $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{M}'}$ for all $a_1, \dots, a_n \in |\mathfrak{M}|$.

Remark 1. If the language contains no constant or **function symbols**, then any $N \subseteq |\mathfrak{M}|$ determines a **substructure** \mathfrak{N} of \mathfrak{M} with **domain** $|\mathfrak{N}| = N$ by putting $R^{\mathfrak{N}} = R^{\mathfrak{M}} \cap N^n$. mod:bas:sub:
rem:substructure

bas.3 Overspill

Theorem bas.5. *If a set Γ of sentences has arbitrarily large finite models, then it has an infinite model.* mod:bas:ove:
sec

Proof. Expand the language of Γ by adding countably many new constants c_0, c_1, \dots and consider the set $\Gamma \cup \{c_i \neq c_j : i \neq j\}$. To say that Γ has arbitrarily large finite models means that for every $m > 0$ there is $n \geq m$ such that Γ has a model of cardinality n . This implies that $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ is finitely satisfiable. By compactness, $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ has a model \mathfrak{M} whose domain must be infinite, since it satisfies all inequalities $c_i \neq c_j$. \square

Proposition bas.6. *There is no sentence φ of any first-order language that is true in a **structure** \mathfrak{M} if and only if the domain $|\mathfrak{M}|$ of the **structure** is infinite.* mod:bas:ove:
inf-not-fo

Proof. If there were such a φ , its negation $\neg\varphi$ would be true in all and only the finite **structures**, and it would therefore have arbitrarily large finite models but it would lack an infinite model, contradicting **Theorem bas.5**. \square

bas.4 Isomorphic Structures

mod:bas:iso:
sec

First-order **structures** can be alike in one of two ways. One way in which they can be alike is that they make the same **sentences** true. We call such **structures** *elementarily equivalent*. But structures can be very different and still make the same **sentences** true—for instance, one can be **enumerable** and the other not. This is because there are lots of features of a **structure** that cannot be expressed in first-order languages, either because the language is not rich enough, or because of fundamental limitations of first-order logic such as the Löwenheim-Skolem theorem. So another, stricter, aspect in which **structures** can be alike is if they are fundamentally the same, in the sense that they only differ in the objects that make them up, but not in their structural features. A way of making this precise is by the notion of an *isomorphism*.

mod:bas:iso:
defn:elem-equiv

Definition bas.7. Given two **structures** \mathfrak{M} and \mathfrak{M}' for the same **language** \mathcal{L} , we say that \mathfrak{M} is *elementarily equivalent* to \mathfrak{M}' , written $\mathfrak{M} \equiv \mathfrak{M}'$, if and only if for every **sentence** φ of \mathcal{L} , $\mathfrak{M} \models \varphi$ iff $\mathfrak{M}' \models \varphi$.

mod:bas:iso:
defn:isomorphism

Definition bas.8. Given two **structures** \mathfrak{M} and \mathfrak{M}' for the same **language** \mathcal{L} , we say that \mathfrak{M} is *isomorphic* to \mathfrak{M}' , written $\mathfrak{M} \simeq \mathfrak{M}'$, if and only if there is a function $h: |\mathfrak{M}| \rightarrow |\mathfrak{M}'|$ such that:

1. h is **injective**: if $h(x) = h(y)$ then $x = y$;
2. h is **surjective**: for every $y \in |\mathfrak{M}'|$ there is $x \in |\mathfrak{M}|$ such that $h(x) = y$;
3. for every **constant symbol** c : $h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$;
4. for every n -place **predicate symbol** P :

mod:bas:iso:
defn:iso-const
mod:bas:iso:
defn:iso-pred

$$\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}} \quad \text{iff} \quad \langle h(a_1), \dots, h(a_n) \rangle \in P^{\mathfrak{M}'};$$

mod:bas:iso:
defn:iso-func

5. for every n -place **function symbol** f :

$$h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{M}'}(h(a_1), \dots, h(a_n)).$$

mod:bas:iso:
thm:isom

Theorem bas.9. *If $\mathfrak{M} \simeq \mathfrak{M}'$ then $\mathfrak{M} \equiv \mathfrak{M}'$.*

Proof. Let h be an isomorphism of \mathfrak{M} onto \mathfrak{M}' . For any assignment s , $h \circ s$ is the composition of h and s , i.e., the assignment in \mathfrak{M}' such that $(h \circ s)(x) = h(s(x))$. By induction on t and φ one can prove the stronger claims:

- a. $h(\text{Val}_s^{\mathfrak{M}}(t)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(t)$.
- b. $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M}', h \circ s \models \varphi$.

The first is proved by induction on the complexity of t .

1. If $t \equiv c$, then $\text{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}}$ and $\text{Val}_{h \circ s}^{\mathfrak{M}'}(c) = c^{\mathfrak{M}'}$. Thus, $h(\text{Val}_s^{\mathfrak{M}}(t)) = h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ (by (3) of **Definition bas.8**) = $\text{Val}_{h \circ s}^{\mathfrak{M}'}(t)$.

2. If $t \equiv x$, then $\text{Val}_s^{\mathfrak{M}}(x) = s(x)$ and $\text{Val}_{h \circ s}^{\mathfrak{M}'}(x) = h(s(x))$. Thus, $h(\text{Val}_s^{\mathfrak{M}}(x)) = h(s(x)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(x)$.
3. If $t \equiv f(t_1, \dots, t_n)$, then

$$\begin{aligned} \text{Val}_s^{\mathfrak{M}}(t) &= f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)) \quad \text{and} \\ \text{Val}_{h \circ s}^{\mathfrak{M}'}(t) &= f^{\mathfrak{M}'}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)). \end{aligned}$$

The induction hypothesis is that for each i , $h(\text{Val}_s^{\mathfrak{M}}(t_i)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_i)$. So,

$$\begin{aligned} h(\text{Val}_s^{\mathfrak{M}}(t)) &= h(f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n))) \\ &= h(f^{\mathfrak{M}}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n))) && \text{(bas.1) } \text{mod:bas:iso:} \\ &= f^{\mathfrak{M}'}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)) && \text{(bas.2) } \text{iso-1} \\ &= \text{Val}_{h \circ s}^{\mathfrak{M}'}(t) && \text{mod:bas:iso:} \\ & && \text{iso-2} \end{aligned}$$

Here, eq. (bas.1) follows by induction hypothesis and eq. (bas.2) by (5) of Definition bas.8.

Part (2) is left as an exercise.

If φ is a sentence, the assignments s and $h \circ s$ are irrelevant, and we have $\mathfrak{M} \models \varphi$ iff $\mathfrak{M}' \models \varphi$. \square

Problem bas.2. Carry out the proof of (b) of Theorem bas.9 in detail. Make sure to note where each of the five properties characterizing isomorphisms of Definition bas.8 is used.

Definition bas.10. An *automorphism* of a structure \mathfrak{M} is an isomorphism of \mathfrak{M} onto itself.

Problem bas.3. Show that for any structure \mathfrak{M} , if X is a definable subset of \mathfrak{M} , and h is an automorphism of \mathfrak{M} , then $X = \{h(x) : x \in X\}$ (i.e., X is fixed under h).

bas.5 The Theory of a Structure

Every structure \mathfrak{M} makes some sentences true, and some false. The set of all the sentences it makes true is called its *theory*. That set is in fact a theory, since anything it entails must be true in all its models, including \mathfrak{M} .

Definition bas.11. Given a structure \mathfrak{M} , the *theory* of \mathfrak{M} is the set $\text{Th}(\mathfrak{M})$ of sentences that are true in \mathfrak{M} , i.e., $\text{Th}(\mathfrak{M}) = \{\varphi : \mathfrak{M} \models \varphi\}$.

We also use the term “theory” informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

Proposition bas.12. For any \mathfrak{M} , $\text{Th}(\mathfrak{M})$ is complete.

Proof. For any **sentence** φ either $\mathfrak{M} \models \varphi$ or $\mathfrak{M} \models \neg\varphi$, so either $\varphi \in \text{Th}(\mathfrak{M})$ or $\neg\varphi \in \text{Th}(\mathfrak{M})$. \square

mod:bas:thm:
prop:equiv **Proposition bas.13.** *If $\mathfrak{N} \models \varphi$ for every $\varphi \in \text{Th}(\mathfrak{M})$, then $\mathfrak{M} \equiv \mathfrak{N}$.*

Proof. Since $\mathfrak{N} \models \varphi$ for all $\varphi \in \text{Th}(\mathfrak{M})$, $\text{Th}(\mathfrak{M}) \subseteq \text{Th}(\mathfrak{N})$. If $\mathfrak{N} \models \varphi$, then $\mathfrak{N} \not\models \neg\varphi$, so $\neg\varphi \notin \text{Th}(\mathfrak{M})$. Since $\text{Th}(\mathfrak{M})$ is complete, $\varphi \in \text{Th}(\mathfrak{M})$. So, $\text{Th}(\mathfrak{N}) \subseteq \text{Th}(\mathfrak{M})$, and we have $\mathfrak{M} \equiv \mathfrak{N}$. \square

mod:bas:thm:
remark:R *Remark 2.* Consider $\mathfrak{R} = \langle \mathbb{R}, < \rangle$, the **structure** whose domain is the set \mathbb{R} of the real numbers, in the **language** comprising only a 2-place **predicate symbol** interpreted as the $<$ relation over the reals. Clearly \mathfrak{R} is **non-enumerable**; however, since $\text{Th}(\mathfrak{R})$ is obviously consistent, by the Löwenheim-Skolem theorem it has an **enumerable** model, say \mathfrak{S} , and by **Proposition bas.13**, $\mathfrak{R} \equiv \mathfrak{S}$. Moreover, since \mathfrak{R} and \mathfrak{S} are not isomorphic, this shows that the converse of **Theorem bas.9** fails in general.

bas.6 Partial Isomorphisms

Definition bas.14. Given two **structures** \mathfrak{M} and \mathfrak{N} , a *partial isomorphism* from \mathfrak{M} to \mathfrak{N} is a finite partial function p taking arguments in $|\mathfrak{M}|$ and returning values in $|\mathfrak{N}|$, which satisfies the isomorphism conditions from **Definition bas.8** on its domain:

1. p is **injective**;
2. for every **constant symbol** c : if $p(c^{\mathfrak{M}})$ is defined, then $p(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$;
3. for every n -place **predicate symbol** P : if a_1, \dots, a_n are in the domain of p , then $\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}}$ if and only if $\langle p(a_1), \dots, p(a_n) \rangle \in P^{\mathfrak{N}}$;
4. for every n -place **function symbol** f : if a_1, \dots, a_n are in the domain of p , then $p(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{N}}(p(a_1), \dots, p(a_n))$.

That p is finite means that $\text{dom}(p)$ is finite.

Notice that the empty function \emptyset is always a partial isomorphism between any two **structures**.

mod:bas:pis:
defn:partialisom **Definition bas.15.** Two **structures** \mathfrak{M} and \mathfrak{N} , are *partially isomorphic*, written $\mathfrak{M} \simeq_p \mathfrak{N}$, if and only if there is a non-empty set I of partial isomorphisms between \mathfrak{M} and \mathfrak{N} satisfying the *back-and-forth* property:

1. (*Forth*) For every $p \in I$ and $a \in |\mathfrak{M}|$ there is $q \in I$ such that $p \subseteq q$ and a is in the domain of q ;
2. (*Back*) For every $p \in I$ and $b \in |\mathfrak{N}|$ there is $q \in I$ such that $p \subseteq q$ and b is in the range of q .

mod:bas:pis:
thm:p-isom1 **Theorem bas.16.** *If $\mathfrak{M} \simeq_p \mathfrak{N}$ and \mathfrak{M} and \mathfrak{N} are **enumerable**, then $\mathfrak{M} \simeq \mathfrak{N}$.*

Proof. Since \mathfrak{M} and \mathfrak{N} are **enumerable**, let $|\mathfrak{M}| = \{a_0, a_1, \dots\}$ and $|\mathfrak{N}| = \{b_0, b_1, \dots\}$. Starting with an arbitrary $p_0 \in I$, we define an increasing sequence of partial isomorphisms $p_0 \subseteq p_1 \subseteq p_2 \subseteq \dots$ as follows:

1. if $n + 1$ is odd, say $n = 2r$, then using the Forth property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and a_r is in the domain of p_{n+1} ;
2. if $n + 1$ is even, say $n + 1 = 2r$, then using the Back property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and b_r is in the range of p_{n+1} .

If we now put:

$$p = \bigcup_{n \geq 0} p_n,$$

we have that p is a an isomorphism between \mathfrak{M} and \mathfrak{N} . □

Problem bas.4. Show in detail that p as defined in [Theorem bas.16](#) is in fact an isomorphism.

Theorem bas.17. *Suppose \mathfrak{M} and \mathfrak{N} are structures for a purely relational language (a language containing only predicate symbols, and no function symbols or constants). Then if $\mathfrak{M} \simeq_p \mathfrak{N}$, also $\mathfrak{M} \equiv \mathfrak{N}$.* [mod:bas:pis:](#)
[thm:p-isom2](#)

Proof. By induction on **formulas**, one shows that if a_1, \dots, a_n and b_1, \dots, b_n are such that there is a partial isomorphism p mapping each a_i to b_i and $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ (for $i = 1, \dots, n$), then $\mathfrak{M}, s_1 \models \varphi$ if and only if $\mathfrak{N}, s_2 \models \varphi$. The case for $n = 0$ gives $\mathfrak{M} \equiv \mathfrak{N}$. □

Remark 3. If **function symbols** are present, the previous result is still true, but one needs to consider the isomorphism induced by p between the **substructure** of \mathfrak{M} generated by a_1, \dots, a_n and the **substructure** of \mathfrak{N} generated by b_1, \dots, b_n .

The previous result can be “broken down” into stages by establishing a connection between the number of nested quantifiers in a **formula** and how many times the relevant partial isomorphisms can be extended.

Definition bas.18. For any **formula** φ , the *quantifier rank* of φ , denoted by $\text{qr}(\varphi) \in \mathbb{N}$, is recursively defined as the highest number of nested quantifiers in φ . Two **structures** \mathfrak{M} and \mathfrak{N} are *n-equivalent*, written $\mathfrak{M} \equiv_n \mathfrak{N}$, if they agree on all sentences of quantifier rank less than or equal to n .

Proposition bas.19. *Let \mathcal{L} be a finite purely relational language, i.e., a language containing finitely many predicate symbols and constant symbols, and no function symbols. Then for each $n \in \mathbb{N}$ there are only finitely many first-order sentences in the language \mathcal{L} that have quantifier rank no greater than n , up to logical equivalence.* [mod:bas:pis:](#)
[prop:qr-finite](#)

Proof. By induction on n . □

Definition bas.20. Given a **structure** \mathfrak{M} , let $|\mathfrak{M}|^{<\omega}$ be the set of all finite sequences over $|\mathfrak{M}|$. We use $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ to range over finite sequences of elements. If $\mathbf{a} \in |\mathfrak{M}|^{<\omega}$ and $a \in |\mathfrak{M}|$, then $\mathbf{a}a$ represents the *concatenation* of \mathbf{a} with a .

Definition bas.21. Given **structures** \mathfrak{M} and \mathfrak{N} , we define relations $I_n \subseteq |\mathfrak{M}|^{<\omega} \times |\mathfrak{N}|^{<\omega}$ between sequences of equal length, by recursion on n as follows:

1. $I_0(\mathbf{a}, \mathbf{b})$ if and only if \mathbf{a} and \mathbf{b} satisfy the same atomic **formulas** in \mathfrak{M} and \mathfrak{N} ; i.e., if $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ and φ is atomic with all **variables** among x_1, \dots, x_n , then $\mathfrak{M}, s_1 \models \varphi$ if and only if $\mathfrak{N}, s_2 \models \varphi$.
2. $I_{n+1}(\mathbf{a}, \mathbf{b})$ if and only if for every $a \in A$ there is a $b \in B$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and vice-versa.

Definition bas.22. Write $\mathfrak{M} \approx_n \mathfrak{N}$ if $I_n(\emptyset, \emptyset)$ holds of \mathfrak{M} and \mathfrak{N} (where \emptyset is the empty sequence).

mod:bas:pis: **Theorem bas.23.** *Let \mathcal{L} be a purely relational **language**. Then $I_n(\mathbf{a}, \mathbf{b})$ implies that for every φ such that $\text{qr}(\varphi) \leq n$, we have $\mathfrak{M}, \mathbf{a} \models \varphi$ if and only if $\mathfrak{N}, \mathbf{b} \models \varphi$ (where again \mathbf{a} satisfies φ if any s such that $s(x_i) = a_i$ satisfies φ). Moreover, if \mathcal{L} is finite, the converse also holds.*

Proof. The proof that $I_n(\mathbf{a}, \mathbf{b})$ implies that \mathbf{a} and \mathbf{b} satisfy the same **formulas** of quantifier rank no greater than n is by an easy induction on φ . For the converse we proceed by induction on n , using [Proposition bas.19](#), which ensures that for each n there are at most finitely many non-equivalent **formulas** of that quantifier rank.

For $n = 0$ the hypothesis that \mathbf{a} and \mathbf{b} satisfy the same quantifier-free **formulas** gives that they satisfy the same atomic ones, so that $I_0(\mathbf{a}, \mathbf{b})$.

For the $n + 1$ case, suppose that \mathbf{a} and \mathbf{b} satisfy the same **formulas** of quantifier rank no greater than $n + 1$; in order to show that $I_{n+1}(\mathbf{a}, \mathbf{b})$ suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and by the inductive hypothesis again suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same **formulas** of quantifier rank no greater than n .

Given $a \in |\mathfrak{M}|$, let τ_n^a be set of **formulas** $\psi(x, \mathbf{y})$ of rank no greater than n satisfied by $\mathbf{a}a$ in \mathfrak{M} ; τ_n^a is finite, so we can assume it is a single first-order **formula**. It follows that \mathbf{a} satisfies $\exists x \tau_n^a(x, \mathbf{y})$, which has quantifier rank no greater than $n + 1$. By hypothesis \mathbf{b} satisfies the same **formula** in \mathfrak{N} , so that there is a $b \in |\mathfrak{N}|$ such that $\mathbf{b}b$ satisfies τ_n^a ; in particular, $\mathbf{b}b$ satisfies the same **formulas** of quantifier rank no greater than n as $\mathbf{a}a$. Similarly one shows that for every $b \in |\mathfrak{N}|$ there is a $a \in |\mathfrak{M}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same **formulas** of quantifier rank no greater than n , which completes the proof. \square

mod:bas:pis: **Corollary bas.24.** *If \mathfrak{M} and \mathfrak{N} are purely relational **structures** in a finite **language**, then $\mathfrak{M} \approx_n \mathfrak{N}$ if and only if $\mathfrak{M} \equiv_n \mathfrak{N}$. In particular $\mathfrak{M} \equiv \mathfrak{N}$ if and only if for each n , $\mathfrak{M} \approx_n \mathfrak{N}$.*

bas.7 Dense Linear Orders

Definition bas.25. A dense linear ordering without endpoints is a structure \mathfrak{M} for the language containing a single 2-place predicate symbol $<$ satisfying the following sentences:

1. $\forall x x < x$;
2. $\forall x \forall y \forall z (x < y \rightarrow (y < z \rightarrow x < z))$;
3. $\forall x \forall y (x < y \vee x = y \vee y < x)$;
4. $\forall x \exists y x < y$;
5. $\forall x \exists y y < x$;
6. $\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y))$.

Theorem bas.26. Any two enumerable dense linear orderings without endpoints are isomorphic.

*mod:bas:dlo:
thm:cantorQ*

Proof. Let \mathfrak{M}_1 and \mathfrak{M}_2 be enumerable dense linear orderings without endpoints, with $<_1 = <^{\mathfrak{M}_1}$ and $<_2 = <^{\mathfrak{M}_2}$, and let \mathcal{I} be the set of all partial isomorphisms between them. \mathcal{I} is not empty since at least $\emptyset \in \mathcal{I}$. We show that \mathcal{I} satisfies the Back-and-Forth property. Then $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$, and the theorem follows by Theorem bas.16.

To show \mathcal{I} satisfies the Forth property, let $p \in \mathcal{I}$ and let $p(a_i) = b_i$ for $i = 1, \dots, n$, and without loss of generality suppose $a_1 <_1 a_2 <_1 \dots <_1 a_n$. Given $a \in |\mathfrak{M}_1|$, find $b \in |\mathfrak{M}_2|$ as follows:

1. if $a <_2 a_1$ let $b \in |\mathfrak{M}_2|$ be such that $b <_2 b_1$;
2. if $a_n <_1 a$ let $b \in |\mathfrak{M}_2|$ be such that $b_n <_2 b$;
3. if $a_i <_1 a <_1 a_{i+1}$ for some i , then let $b \in |\mathfrak{M}_2|$ be such that $b_i <_2 b <_2 b_{i+1}$.

It is always possible to find a b with the desired property since \mathfrak{M}_2 is a dense linear ordering without endpoints. Define $q = p \cup \{(a, b)\}$ so that $q \in \mathcal{I}$ is the desired extension of p . This establishes the Forth property. The Back property is similar. So $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$; by Theorem bas.16, $\mathfrak{M}_1 \simeq \mathfrak{M}_2$. \square

Problem bas.5. Complete the proof of Theorem bas.26 by verifying that \mathcal{I} satisfies the Back property.

Remark 4. Let \mathfrak{S} be any enumerable dense linear ordering without endpoints. Then (by Theorem bas.26) $\mathfrak{S} \simeq \mathfrak{Q}$, where $\mathfrak{Q} = (\mathbb{Q}, <)$ is the enumerable dense linear ordering having the set \mathbb{Q} of the rational numbers as its domain. Now consider again the structure $\mathfrak{R} = (\mathbb{R}, <)$ from Remark 2. We saw that there is an enumerable structure \mathfrak{S} such that $\mathfrak{R} \equiv \mathfrak{S}$. But \mathfrak{S} is an enumerable dense linear ordering without endpoints, and so it is isomorphic (and hence

elementarily equivalent) to the **structure** \mathfrak{Q} . By transitivity of elementary equivalence, $\mathfrak{R} \equiv \mathfrak{Q}$. (We could have shown this directly by establishing $\mathfrak{R} \simeq_p \mathfrak{Q}$ by the same back-and-forth argument.)

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Bibliography