Chapter udf

Basics of Model Theory

bas.1 Reducts and Expansions

Often it is useful or necessary to compare languages which have symbols in common, as well as structures for these languages. The most comon case is when all the symbols in a language $\mathcal{L}$ are also part of a language $\mathcal{L}'$, i.e., $\mathcal{L} \subseteq \mathcal{L}'$. An $\mathcal{L}$-structure $\mathfrak{M}$ can then always be expanded to an $\mathcal{L}'$-structure by adding interpretations of the additional symbols while leaving the interpretations of the common symbols the same. On the other hand, from an $\mathcal{L}'$-structure $\mathfrak{M}'$ we can obtain an $\mathcal{L}$-structure simpy by "forgetting" the interpretations of the symbols that do not occur in $\mathcal{L}$.

mod:bas:red:defn:reduct **Definition bas.1.** Suppose $\mathcal{L} \subseteq \mathcal{L}'$, $\mathfrak{M}$ is an $\mathcal{L}$-structure and $\mathfrak{M}'$ is an $\mathcal{L}'$-structure. $\mathfrak{M}$ is the *reduct* of $\mathfrak{M}'$ to $\mathcal{L}$, and $\mathfrak{M}'$ is an *expansion* of $\mathfrak{M}$ to $\mathcal{L}'$ iff

1. $|\mathfrak{M}| = |\mathfrak{M}'|$
2. For every constant symbol $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$.
3. For every function symbol $f \in \mathcal{L}$, $f^{\mathfrak{M}} = f^{\mathfrak{M}'}$.
4. For every predicate symbol $P \in \mathcal{L}$, $P^{\mathfrak{M}} = P^{\mathfrak{M}'}$.

mod:bas:red:prop:reduct **Proposition bas.2.** *If an $\mathcal{L}$-structure $\mathfrak{M}$ is a reduct of an $\mathcal{L}'$-structure $\mathfrak{M}'$, then for all $\mathcal{L}$-sentences φ,*

$$\mathfrak{M} \vDash \varphi \text{ iff } \mathfrak{M}' \vDash \varphi.$$

Proof. Exercise. □

Problem bas.1. Prove Proposition bas.2.

Definition bas.3. When we have an $\mathcal{L}$-structure $\mathfrak{M}$, and $\mathcal{L}' = \mathcal{L} \cup \{P\}$ is the expansion of $\mathcal{L}$ obtained by adding a single n-place predicate symbol P, and $R \subseteq |\mathfrak{M}|^n$ is an n-place relation, then we write $(\mathfrak{M}, R)$ for the expansion $\mathfrak{M}'$ of $\mathfrak{M}$ with $P^{\mathfrak{M}'} = R$.

bas.2 Substructures

The domain of a structure $\mathfrak{M}$ may be a subset of another $\mathfrak{M}'$. But we should obviously only consider $\mathfrak{M}$ a "part" of $\mathfrak{M}'$ if not only $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$, but $\mathfrak{M}$ and $\mathfrak{M}'$ "agree" in how they interpret the symbols of the language at least on the shared part $|\mathfrak{M}|$. mod:bas:sub: sec

Definition bas.4. Given structures $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is a *substructure* of $\mathfrak{M}'$, and $\mathfrak{M}'$ an *extension* of $\mathfrak{M}$, written $\mathfrak{M} \subseteq \mathfrak{M}'$, iff mod:bas:sub: defn:substructure

1. $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$,
2. For each constant $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$;
3. For each n-place predicate symbol $f \in \mathcal{L}$ $f^{\mathfrak{M}}(a_1, \dots, a_n) = f^{\mathfrak{M}'}(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in |\mathfrak{M}|$.
4. For each n-place predicate symbol $R \in \mathcal{L}$, $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{M}}$ iff $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{M}'}$ for all $a_1, \dots, a_n \in |\mathfrak{M}|$.

Remark 1. If the language contains no constant or function symbols, then any $N \subseteq |\mathfrak{M}|$ determines a substructure $\mathfrak{N}$ of $\mathfrak{M}$ with domain $|\mathfrak{N}| = N$ by putting $R^{\mathfrak{N}} = R^{\mathfrak{M}} \cap N^n$. mod:bas:sub: rem:substructure

bas.3 Overspill

mod:bas:ove: sec

Theorem bas.5. *If a set Γ of sentences has arbitrarily large finite models, then it has an infinite model.* *mod:bas:ove: overspill*

Proof. Expand the language of Γ by adding countably many new constants c_0, c_1, ... and consider the set $\Gamma \cup \{c_i \neq c_j : i \neq j\}$. To say that Γ has arbitrarily large finite models means that for every $m > 0$ there is $n \geq m$ such that Γ has a model of cardinality n. This implies that $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ is finitely satisfiable. By compactness, $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ has a model $\mathfrak{M}$ whose domain must be infinite, since it satisfies all inequalities $c_i \neq c_j$. □

Proposition bas.6. *There is no sentence φ of any first-order language that is true in a structure $\mathfrak{M}$ if and only if the domain $|\mathfrak{M}|$ of the structure is infinite.* *mod:bas:ove: inf-not-fo*

Proof. If there were such a φ, its negation $\neg\varphi$ would be true in all and only the finite structures, and it would therefore have arbitrarily large finite models but it would lack an infinite model, contradicting Theorem bas.5. □

bas.4 Isomorphic Structures

mod:bas:iso:sec

First-order structures can be alike in one of two ways. One way in which the can be alike is that they make the same sentences true. We call such structures *elementarily equivalent*. But structures can be very different and still make the same sentences true—for instance, one can be enumerable and the other not. This is because there are lots of features of a structure that cannot be expressed in first-order languages, either because the language is not rich enough, or because of fundamental limitations of first-order logic such as the Löwenheim-Skolem theorem. So another, stricter, aspect in which structures can be alike is if they are fundamentally the same, in the sense that they only differ in the objects that make them up, but not in their structural features. A way of making this precise is by the notion of an *isomorphism*.

mod:bas:iso:defn:elem-equiv

Definition bas.7. Given two structures $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is *elementarily equivalent to* $\mathfrak{M}'$, written $\mathfrak{M} \equiv \mathfrak{M}'$, if and only if for every sentence φ of $\mathcal{L}$, $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$.

mod:bas:iso:defn:isomorphism

Definition bas.8. Given two structures $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is *isomorphic to* $\mathfrak{M}'$, written $\mathfrak{M} \simeq \mathfrak{M}'$, if and only if there is a function $h\colon |\mathfrak{M}| \to |\mathfrak{M}'|$ such that:

1. h is injective: if $h(x) = h(y)$ then $x = y$;
2. h is surjective: for every $y \in |\mathfrak{M}'|$ there is $x \in |\mathfrak{M}|$ such that $h(x) = y$;
3. mod:bas:iso:defn:iso-const for every constant symbol c: $h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$;
4. mod:bas:iso:defn:iso-pred for every n-place predicate symbol P:
$$\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}} \quad \text{iff} \quad \langle h(a_1), \dots, h(a_n) \rangle \in P^{\mathfrak{M}'};$$
5. mod:bas:iso:defn:iso-func for every n-place function symbol f:
$$h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{M}'}(h(a_1), \dots, h(a_n)).$$

mod:bas:iso:thm:isom

Theorem bas.9. *If $\mathfrak{M} \simeq \mathfrak{M}'$ then $\mathfrak{M} \equiv \mathfrak{M}'$.*

Proof. Let h be an isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$. For any assignment s, $h \circ s$ is the composition of h and s, i.e., the assignment in $\mathfrak{M}'$ such that $(h \circ s)(x) = h(s(x))$. By induction on t and φ one can prove the stronger claims:

a. $h(\mathrm{Val}^{\mathfrak{M}}_s(t)) = \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t)$.

b. $\mathfrak{M}, s \vDash \varphi$ iff $\mathfrak{M}', h \circ s \vDash \varphi$.

The first is proved by induction on the complexity of t.

1. If $t \equiv c$, then $\mathrm{Val}^{\mathfrak{M}}_s(c) = c^{\mathfrak{M}}$ and $\mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(c) = c^{\mathfrak{M}'}$. Thus, $h(\mathrm{Val}^{\mathfrak{M}}_s(t)) = h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ (by (3) of Definition bas.8) $= \mathrm{Val}^{\mathfrak{M}'}_{h\circ s}(t)$.

2. If $t \equiv x$, then $\mathrm{Val}^{\mathfrak{M}}_{s}(x) = s(x)$ and $\mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(x) = h(s(x))$. Thus, $h(\mathrm{Val}^{\mathfrak{M}}_{s}(x)) = h(s(x)) = \mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(x)$.

3. If $t \equiv f(t_1, \dots, t_n)$, then

$$\begin{aligned} \mathrm{Val}^{\mathfrak{M}}_{s}(t) &= f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}}_{s}(t_n)) \quad \text{and} \\ \mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t) &= f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t_n)). \end{aligned}$$

   The induction hypothesis is that for each i, $h(\mathrm{Val}^{\mathfrak{M}}_{s}(t_i)) = \mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t_i)$. So,

$$\begin{aligned} h(\mathrm{Val}^{\mathfrak{M}}_{s}(t)) &= h(f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}}_{s}(t_n))) \\ &= h(f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t_n))) && \text{(bas.1)} \\ &= f^{\mathfrak{M}'}(\mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t_n)) && \text{(bas.2)} \\ &= \mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t) \end{aligned}$$

mod:bas:iso:iso-1
mod:bas:iso:iso-2

   Here, eq. (bas.1) follows by induction hypothesis and eq. (bas.2) by (5) of Definition bas.8.

Part (2) is left as an exercise.

If φ is a sentence, the assignments s and $h \circ s$ are irrelevant, and we have $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$. □

Problem bas.2. Carry out the proof of (b) of Theorem bas.9 in detail. Make sure to note where each of the five properties characterizing isomorphisms of Definition bas.8 is used.

Definition bas.10. An *automorphism* of a structure $\mathfrak{M}$ is an isomorphism of $\mathfrak{M}$ onto itself.

Problem bas.3. Show that for any structure $\mathfrak{M}$, if X is a definable subset of $\mathfrak{M}$, and h is an automorphism of $\mathfrak{M}$, then $X = \{h(x) : x \in X\}$ (i.e., X is fixed under h).

bas.5 The Theory of a Structure

Every structure $\mathfrak{M}$ makes some sentences true, and some false. The set of all the sentences it makes true is called its *theory*. That set is in fact a theory, since anything it entails must be true in all its models, including $\mathfrak{M}$.

Definition bas.11. Given a structure $\mathfrak{M}$, the *theory* of $\mathfrak{M}$ is the set $\mathrm{Th}(\mathfrak{M})$ of sentences that are true in $\mathfrak{M}$, i.e., $\mathrm{Th}(\mathfrak{M}) = \{\varphi : \mathfrak{M} \vDash \varphi\}$.

We also use the term "theory" informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

Proposition bas.12. *For any $\mathfrak{M}$, $\mathrm{Th}(\mathfrak{M})$ is complete.*

Proof. For any sentence φ either $\mathfrak{M} \models \varphi$ or $\mathfrak{M} \models \neg\varphi$, so either $\varphi \in \mathrm{Th}(\mathfrak{M})$ or $\neg\varphi \in \mathrm{Th}(\mathfrak{M})$. □

mod:bas:thm:prop:equiv

Proposition bas.13. *If $\mathfrak{N} \models \varphi$ for every $\varphi \in \mathrm{Th}(\mathfrak{M})$, then $\mathfrak{M} \equiv \mathfrak{N}$.*

Proof. Since $\mathfrak{N} \models \varphi$ for all $\varphi \in \mathrm{Th}(\mathfrak{M})$, $\mathrm{Th}(\mathfrak{M}) \subseteq \mathrm{Th}(\mathfrak{N})$. If $\mathfrak{N} \models \varphi$, then $\mathfrak{N} \not\models \neg\varphi$, so $\neg\varphi \notin \mathrm{Th}(\mathfrak{M})$. Since $\mathrm{Th}(\mathfrak{M})$ is complete, $\varphi \in \mathrm{Th}(\mathfrak{M})$. So, $\mathrm{Th}(\mathfrak{N}) \subseteq \mathrm{Th}(\mathfrak{M})$, and we have $\mathfrak{M} \equiv \mathfrak{N}$. □

mod:bas:thm:remark:R

Remark 2. Consider $\mathfrak{R} = \langle \mathbb{R}, < \rangle$, the structure whose domain is the set $\mathbb{R}$ of the real numbers, in the language comprising only a 2-place predicate symbol interpreted as the $<$ relation over the reals. Clearly $\mathfrak{R}$ is non-enumerable; however, since $\mathrm{Th}(\mathfrak{R})$ is obviously consistent, by the Löwenheim-Skolem theorem it has an enumerable model, say $\mathfrak{S}$, and by Proposition bas.13, $\mathfrak{R} \equiv \mathfrak{S}$. Moreover, since $\mathfrak{R}$ and $\mathfrak{S}$ are not isomorphic, this shows that the converse of Theorem bas.9 fails in general.

bas.6 Partial Isomorphisms

Definition bas.14. Given two structures $\mathfrak{M}$ and $\mathfrak{N}$, a *partial isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ is a finite partial function p taking arguments in $|\mathfrak{M}|$ and returning values in $|\mathfrak{N}|$, which satisfies the isomorphism conditions from Definition bas.8 on its domain:

1. p is injective;
2. for every constant symbol c: if $p(c^{\mathfrak{M}})$ is defined, then $p(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$;
3. for every n-place predicate symbol P: if $a_1, \ldots, a_n$ are in the domain of p, then $\langle a_1, \ldots, a_n \rangle \in P^{\mathfrak{M}}$ if and only if $\langle p(a_1), \ldots, p(a_n) \rangle \in P^{\mathfrak{N}}$;
4. for every n-place function symbol f: if $a_1, \ldots, a_n$ are in the domain of p, then $p(f^{\mathfrak{M}}(a_1, \ldots, a_n)) = f^{\mathfrak{N}}(p(a_1),\ dots, p(a_n))$.

That p is finite means that $\mathrm{dom}(p)$ is finite.

Notice that the empty function $\emptyset$ is always a partial isomorphism between any two structures.

mod:bas:pis:defn:partialisom

Definition bas.15. Two structures $\mathfrak{M}$ and $\mathfrak{N}$, are *partially isomorphic*, written $\mathfrak{M} \simeq_p \mathfrak{N}$, if and only if there is a non-empty set I of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ satisfying the *back-and-forth* property:

1. (*Forth*) For every $p \in I$ and $a \in |\mathfrak{M}|$ there is $q \in I$ such that $p \subseteq q$ and a is in the domain of q;
2. (*Back*) For every $p \in I$ and $b \in |\mathfrak{N}|$ there is $q \in I$ such that $p \subseteq q$ and b is in the range of q.

mod:bas:pis:thm:p-isom1

Theorem bas.16. *If $\mathfrak{M} \simeq_p \mathfrak{N}$ and $\mathfrak{M}$ and $\mathfrak{N}$ are enumerable, then $\mathfrak{M} \simeq \mathfrak{N}$.*

Proof. Since $\mathfrak{M}$ and $\mathfrak{N}$ are enumerable, let $|\mathfrak{M}| = \{a_0, a_1, \dots\}$ and $|\mathfrak{N}| = \{b_0, b_1, \dots\}$. Starting with an arbitrary $p_0 \in I$, we define an increasing sequence of partial isomorphisms $p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$ as follows:

1. if $n+1$ is odd, say $n = 2r$, then using the Forth property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and a_r is in the domain of p_{n+1};
2. if $n+1$ is even, say $n+1 = 2r$, then using the Back property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and b_r is in the range of p_{n+1}.

If we now put:

$$p = \bigcup_{n \geq 0} p_n,$$

we have that p is a an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. □

Problem bas.4. Show in detail that p as defined in Theorem bas.16 is in fact an isomorphism.

Theorem bas.17. *Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are structures for a purely relational language (a language containing only predicate symbols, and no function symbols or constants). Then if $\mathfrak{M} \simeq_p \mathfrak{N}$, also $\mathfrak{M} \equiv \mathfrak{N}$.* *mod:bas:pis:thm:p-isom2*

Proof. By induction on formulas, one shows that if $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are such that there is a partial isomorphism p mapping each a_i to b_i and $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ (for $i = 1, \dots, n$), then $\mathfrak{M}, s_1 \vDash \varphi$ if and only if $\mathfrak{N}, s_2 \vDash \varphi$. The case for $n = 0$ gives $\mathfrak{M} \equiv \mathfrak{N}$. □

Remark 3. If function symbols are present, the previous result is still true, but one needs to consider the isomorphism induced by p between the substructure of $\mathfrak{M}$ generated by $a_1, \dots, a_n$ and the substructure of $\mathfrak{N}$ generated by $b_1, \dots, b_n$.

The previous result can be "broken down" into stages by establishing a connection between the number of nested quantifiers in a formula and how many times the relevant partial isomorphisms can be extended.

Definition bas.18. For any formula φ, the *quantifier rank* of φ, denoted by $\mathrm{qr}(\varphi) \in \mathbb{N}$, is recursively defined as the highest number of nested quantifiers in φ. Two structures $\mathfrak{M}$ and $\mathfrak{N}$ are *n-equivalent*, written $\mathfrak{M} \equiv_n \mathfrak{N}$, if they agree on all sentences of quantifier rank less than or equal to n.

Proposition bas.19. *Let $\mathcal{L}$ be a finite purely relational language, i.e., a language containing finitely many predicate symbols and constant symbols, and no function symbols. Then for each $n \in \mathbb{N}$ there are only finitely many first-order sentences in the language $\mathcal{L}$ that have quantifier rank no greater than n, up to logical equivalence.* *mod:bas:pis:prop:qr-finite*

Proof. By induction on n. □

Definition bas.20. Given a structure $\mathfrak{M}$, let $|\mathfrak{M}|^{<\omega}$ be the set of all finite sequences over $|\mathfrak{M}|$. We use $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ to range over finite sequences of elements. If $\mathbf{a} \in |\mathfrak{M}|^{<\omega}$ and $a \in |\mathfrak{M}|$, then $\mathbf{a}a$ represents the *concatenation* of $\mathbf{a}$ with a.

Definition bas.21. Given structures $\mathfrak{M}$ and $\mathfrak{N}$, we define relations $I_n \subseteq |\mathfrak{M}|^{<\omega} \times |\mathfrak{N}|^{<\omega}$ between sequences of equal length, by recursion on n as follows:

1. $I_0(\mathbf{a}, \mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ satisfy the same atomic formulas in $\mathfrak{M}$ and $\mathfrak{N}$; i.e., if $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ and φ is atomic with all variables among $x_1, \dots, x_n$, then $\mathfrak{M}, s_1 \vDash \varphi$ if and only if $\mathfrak{N}, s_2 \vDash \varphi$.

2. $I_{n+1}(\mathbf{a}, \mathbf{b})$ if and only if for every $a \in A$ there is a $b \in B$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and vice-versa.

Definition bas.22. Write $\mathfrak{M} \approx_n \mathfrak{N}$ if $I_n(\Lambda, \Lambda)$ holds of $\mathfrak{M}$ and $\mathfrak{N}$ (where Λ is the empty sequence).

mod:bas:pis: thm:b-n-f **Theorem bas.23.** *Let $\mathcal{L}$ be a purely relational language. Then $I_n(\mathbf{a}, \mathbf{b})$ implies that for every φ such that $\mathrm{qr}(\varphi) \leq n$, we have $\mathfrak{M}, \mathbf{a} \vDash \varphi$ if and only if $\mathfrak{N}, \mathbf{b} \vDash \varphi$ (where again $\mathbf{a}$ satisfies φ if any s such that $s(x_i) = a_i$ satisfies φ). Moreover, if $\mathcal{L}$ is finite, the converse also holds.*

Proof. The proof that $I_n(\mathbf{a}, \mathbf{b})$ implies that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of quantifier rank no greater than n is by an easy induction on φ. For the converse we proceed by induction on n, using Proposition bas.19, which ensures that for each n there are at most finitely many non-equivalent formulas of that quantifier rank.

For $n = 0$ the hypothesis that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same quantifier-free formulas gives that they satisfy the same atomic ones, so that $I_0(\mathbf{a}, \mathbf{b})$.

For the $n+1$ case, suppose that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of quantifier rank no greater than $n+1$; in order to show that $I_{n+1}(\mathbf{a}, \mathbf{b})$ suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and by the inductive hypothesis again suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n.

Given $a \in |\mathfrak{M}|$, let τ_n^a be set of formulas $\psi(x, \mathbf{y})$ of rank no greater than n satisfied by $\mathbf{a}a$ in $\mathfrak{M}$; τ_n^a is finite, so we can assume it is a single first-order formula. It follows that $\mathbf{a}$ satisfies $\exists x\, \tau_n^a(x, \mathbf{y})$, which has quantifier rank no greater than $n+1$. By hypothesis $\mathbf{b}$ satisfies the same formula in $\mathfrak{N}$, so that there is a $b \in |\mathfrak{N}|$ such that $\mathbf{b}b$ satisfies τ_n^a; in particular, $\mathbf{b}b$ satisfies the same formulas of quantifier rank no greater than n as $\mathbf{a}a$. Similarly one shows that for every $b \in |\mathfrak{N}|$ there is $a \in |\mathfrak{M}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n, which completes the proof. □

mod:bas:pis: cor:b-n-f **Corollary bas.24.** *If $\mathfrak{M}$ and $\mathfrak{N}$ are purely relational structures in a finite language, then $\mathfrak{M} \approx_n \mathfrak{N}$ if and only if $\mathfrak{M} \equiv_n \mathfrak{N}$. In particular $\mathfrak{M} \equiv \mathfrak{N}$ if and only if for each n, $\mathfrak{M} \approx_n \mathfrak{N}$.*

bas.7 Dense Linear Orders

Definition bas.25. A *dense linear ordering without endpoints* is a structure $\mathfrak{M}$ for the language containg a single 2-place predicate symbol $<$ satisfying the following sentences:

1. $\forall x\, x < x$;
2. $\forall x\, \forall y\, \forall z\, (x < y \rightarrow (y < z \rightarrow x < z))$;
3. $\forall x\, \forall y\, (x < y \lor x = y \lor y < x)$;
4. $\forall x\, \exists y\, x < y$;
5. $\forall x\, \exists y\, y < x$;
6. $\forall x\, \forall y\, (x < y \rightarrow \exists z\, (x < z \land z < y))$.

Theorem bas.26. *Any two enumerable dense linear orderings without endpoints are isomorphic.* mod:bas:dlo:thm:cantorQ

Proof. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be enumerable dense linear orderings without endpoints, with $<_1 = <^{\mathfrak{M}_1}$ and $<_2 = <^{\mathfrak{M}_2}$, and let $\mathcal{I}$ be the set of all partial isomorphisms between them. $\mathcal{I}$ is not empty since at least $\emptyset \in \mathcal{I}$. We show that $\mathcal{I}$ satisfies the Back-and-Forth property. Then $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$, and the theorem follows by Theorem bas.16.

To show $\mathcal{I}$ satisifes the Forth property, let $p \in \mathcal{I}$ and let $p(a_i) = b_i$ for $i = 1$, …, n, and without loss of generality suppose $a_1 <_1 a_2 <_1 \dots <_1 a_n$. Given $a \in |\mathfrak{M}_1|$, find $b \in |\mathfrak{M}_2|$ as follows:

1. if $a <_2 a_1$ let $b \in |\mathfrak{M}_2|$ be such that $b <_2 b_1$;
2. if $a_n <_1 a$ let $b \in |\mathfrak{M}_2|$ be such that $b_n <_2 b$;
3. if $a_i <_1 a <_1 a_{i+1}$ for some i, then let $b \in |\mathfrak{M}_2|$ be such that $b_i <_2 b <_2 b_{i+1}$.

It is always possible to find a b with the desired property since $\mathfrak{M}_2$ is a dense linear ordering without endpoints. Define $q = p \cup \{\langle a, b\rangle\}$ so that $q \in \mathcal{I}$ is the desired extension of p. This establishes the Forth property. The Back property is similar. So $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$; by Theorem bas.16, $\mathfrak{M}_1 \simeq \mathfrak{M}_2$. □

Problem bas.5. Complete the proof of Theorem bas.26 by verifying that $\mathcal{I}$ satisfies the Back property.

Remark 4. Let $\mathfrak{S}$ be any enumerable dense linear ordering without endpoints. Then (by Theorem bas.26) $\mathfrak{S} \simeq \mathfrak{Q}$, where $\mathfrak{Q} = (\mathbb{Q}, <)$ is the enumerable dense linear ordering having the set $\mathbb{Q}$ of the rational numbers as its domain. Now consider again the structure $\mathfrak{R} = (\mathbb{R}, <)$ from Remark 2. We saw that there is an enumerable structure $\mathfrak{S}$ such that $\mathfrak{R} \equiv \mathfrak{S}$. But $\mathfrak{S}$ is an enumerable dense linear ordering without endpoints, and so it is isomorphic (and hence elementarily equivalent) to the structure $\mathfrak{Q}$. By transitivity of elementary equivalence, $\mathfrak{R} \equiv \mathfrak{Q}$. (We could have shown this directly by establishing $\mathfrak{R} \simeq_p \mathfrak{Q}$ by the same back-and-forth argument.)

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Bibliography