

## prf.1 Using Definitions

math:prf:def:  
sec

We mentioned that you must be familiar with all definitions that may be used in the proof, and that you can properly apply them. This is a really important point, and it is worth looking at in a bit more detail. Definitions are used to abbreviate properties and relations so we can talk about them more succinctly. The introduced abbreviation is called the *definiendum*, and what it abbreviates is the *definiens*. In proofs, we often have to go back to how the definiendum was introduced, because we have to exploit the logical structure of the definiens (the long version of which the defined term is the abbreviation) to get through our proof. By unpacking definitions, you’re ensuring that you’re getting to the heart of where the logical action is.

We’ll start with an example. Suppose you want to prove the following:

**Proposition prf.1.** *For any sets  $X$  and  $Y$ ,  $X \cup Y = Y \cup X$ .*

In order to even start the proof, we need to know what it means for two sets to be identical; i.e., we need to know what the “=” in that equation means for sets. Sets are defined to be identical whenever they have the same **elements**. So the definition we have to unpack is:

**Definition prf.2.** Sets  $X$  and  $Y$  are *identical*,  $X = Y$ , iff every **element** of  $X$  is an **element** of  $Y$ , and vice versa.

This definition uses  $X$  and  $Y$  as placeholders for arbitrary sets. What it defines—the *definiendum*—is the expression “ $X = Y$ ” by giving the condition under which  $X = Y$  is true. This condition—“every **element** of  $X$  is an **element** of  $Y$ , and vice versa”—is the *definiens*.<sup>1</sup> The definition specifies that  $X = Y$  is true if, and only if (we abbreviate this to “iff”) the condition holds.

When you apply the definition, you have to match the  $X$  and  $Y$  in the definition to the case you’re dealing with. In our case, it means that in order for  $X \cup Y = Y \cup X$  to be true, each  $z \in X \cup Y$  must also be in  $Y \cup X$ , and vice versa. The expression  $X \cup Y$  in the proposition plays the role of  $X$  in the definition, and  $Y \cup X$  that of  $Y$ . Since  $X$  and  $Y$  are used both in the definition and in the statement of the proposition we’re proving, but in different uses, you have to be careful to make sure you don’t mix up the two. For instance, it would be a mistake to think that you could prove the proposition by showing that every **element** of  $X$  is an **element** of  $Y$ , and vice versa—that would show that  $X = Y$ , not that  $X \cup Y = Y \cup X$ . (Also, since  $X$  and  $Y$  may be any two sets, you won’t get very far, because if nothing is assumed about  $X$  and  $Y$  they may well be different sets.)

Within the proof we are dealing with set-theoretic notions such as union, and so we must also know the meanings of the symbol  $\cup$  in order to understand

---

<sup>1</sup>In this particular case—and very confusingly!—when  $X = Y$ , the sets  $X$  and  $Y$  are just one and the same set, even though we use different letters for it on the left and the right side. But the ways in which that set is picked out may be different, and that makes the definition non-trivial.

how the proof should proceed. And sometimes, unpacking the definition gives rise to further definitions to unpack. For instance,  $X \cup Y$  is defined as  $\{z : z \in X \text{ or } z \in Y\}$ . So if you want to prove that  $x \in X \cup Y$ , unpacking the definition of  $\cup$  tells you that you have to prove  $x \in \{z : z \in X \text{ or } z \in Y\}$ . Now you also have to remember that  $x \in \{z : \dots z \dots\}$  iff  $\dots x \dots$ . So, further unpacking the definition of the  $\{z : \dots z \dots\}$  notation, what you have to show is:  $x \in X$  or  $x \in Y$ . So, “every element of  $X \cup Y$  is also an element of  $Y \cup X$ ” really means: “for every  $x$ , if  $x \in X$  or  $x \in Y$ , then  $x \in Y$  or  $x \in X$ .” If we fully unpack the definitions in the proposition, we see that what we have to show is this:

**Proposition prf.3.** *For any sets  $X$  and  $Y$ : (a) for every  $x$ , if  $x \in X$  or  $x \in Y$ , then  $x \in Y$  or  $x \in X$ , and (b) for every  $x$ , if  $x \in Y$  or  $x \in X$ , then  $x \in X$  or  $x \in Y$ .*

What’s important is that unpacking definitions is a necessary part of constructing a proof. Properly doing it is sometimes difficult: you must be careful to distinguish and match the variables in the definition and the terms in the claim you’re proving. In order to be successful, you must know what the question is asking and what all the terms used in the question mean—you will often need to unpack more than one definition. In simple proofs such as the ones below, the solution follows almost immediately from the definitions themselves. Of course, it won’t always be this simple.

**Problem prf.1.** Suppose you are asked to prove that  $X \cap Y \neq \emptyset$ . Unpack all the definitions occurring here, i.e., restate this in a way that does not mention “ $\cap$ ”, “ $=$ ”, or “ $\emptyset$ ”.

## Photo Credits

## Bibliography