prf.1 Reading Proofs

mth:prf:rea:sec Proofs you find in textbooks and articles very seldom give all the details we have so far included in our examples. Authors ofen do not draw attention to when they distinguish cases, when they give an indirect proof, or don't mention that they use a definition. So when you read a proof in a textbook, you will often have to fill in those details for yourself in order to understand the proof. Doing this is also good practice to get the hang of the various moves you have to make in a proof. Let's look at an example.

Proposition prf.1 (Absorption)**.** *For all sets X, Y,*

$$X \cap (X \cup Y) = X$$

Proof. If $z \in X \cap (X \cup Y)$, then $z \in X$, so $X \cap (X \cup Y) \subseteq X$. Now suppose $z \in X$. Then also $z \in X \cup Y$, and therefore also $z \in X \cap (X \cup Y)$. □

The preceding proof of the absorption law is very condensed. There is no mention of any definitions used, no "we have to prove that" before we prove it, etc. Let's unpack it. The proposition proved is a general claim about any sets X and Y, and when the proof mentions X or Y, these are variables for arbitrary sets. The general claims the proof establishes is what's required to prove identity of sets, i.e., that every element of the left side of the identity is an element of the right and vice versa.

> "If $z \in X \cap (X \cup Y)$, then $z \in X$, so $X \cap (X \cup Y) \subseteq X$."

This is the first half of the proof of the identity: it estabishes that if an arbitrary z is an element of the left side, it is also an element of the right, i.e., $X \cap (X \cup Y) \subseteq X$. Assume that $z \in X \cap (X \cup Y)$. Since z is an element of the intersection of two sets iff it is an element of both sets, we can conclude that $z \in X$ and also $z \in X \cup Y$. In particular, $z \in X$, which is what we wanted to show. Since that's all that has to be done for the first half, we know that the rest of the proof must be a proof of the second half, i.e., a proof that $X \subseteq X \cap (X \cup Y)$.

> "Now suppose $z \in X$. Then also $z \in X \cup Y$, and therefore also $z \in X \cap (X \cup Y)$."

We start by assuming that $z \in X$, since we are showing that, for any z, if $z \in X$ then $z \in X \cap (X \cup Y)$. To show that $z \in X \cap (X \cup Y)$, we have to show (by definition of "$\cap$") that (i) $z \in X$ and also (ii) $z \in X \cup Y$. Here (i) is just our assumption, so there is nothing further to prove, and that's why the proof does not mention it again. For (ii), recall that z is an element of a union of sets iff it is an element of at least one of those sets. Since $z \in X$, and $X \cup Y$ is the union of X and Y, this is the case here. So $z \in X \cup Y$. We've shown both (i) $z \in X$ and (ii) $z \in X \cup Y$, hence, by definition of "$\cap$," $z \in X \cap (X \cup Y)$. The proof doesn't mention those definitions; it's assumed the reader has already internalized them. If you haven't, you'll have to go back and remind yourself what they are. Then you'll also have to recognize why it follows from $z \in X$ that $z \in X \cup Y$, and from $z \in X$ and $z \in X \cup Y$ that $z \in X \cap (X \cup Y)$.

Here's another version of the proof above, with everything made explicit:

Proof. [By definition of $=$ for sets, $X \cap (X \cup Y) = X$ we have to show (a) $X \cap (X \cup Y) \subseteq X$ and (b) $X \cap (X \cup Y) \subseteq X$. (a): By definition of $\subseteq$, we have to show that if $z \in X \cap (X \cup Y)$, then $z \in X$.] If $z \in X \cap (X \cup Y)$, then $z \in X$ [since by definition of $\cap$, $z \in X \cap (X \cup Y)$ iff $z \in X$ and $z \in X \cup Y$], so $X \cap (X \cup Y) \subseteq X$. [(b): By definition of $\subseteq$, we have to show that if $z \in X$, then $z \in X \cap (X \cup Y)$.] Now suppose [(1)] $z \in X$. Then also [(2)] $z \in X \cup Y$ [since by (1) $z \in X$ or $z \in Y$, which by definition of $\cup$ means $z \in X \cup Y$], and therefore also $z \in X \cap (X \cup Y)$ [since the definition of $\cap$ requires that $z \in X$, i.e., (1), and $z \in X \cup Y$), i.e., (2)]. □

Problem prf.1. Expand the following proof of $X \cup (X \cap Y) = X$, where you mention all the inference patterns used, why each step follows from assumptions or claims established before it, and where we have to appeal to which definitions.

Proof. If $z \in X \cup (X \cap Y)$ then $z \in X$ or $z \in X \cap Y$. If $z \in X \cap Y$, $z \in X$. Any $z \in X$ is also $\in X \cup (X \cap Y)$. □

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Bibliography