prf.1 Proof by Contradiction

mth:prf:con:sec In the first instance, proof by contradiction is an inference pattern that is used to prove negative claims. Suppose you want to show that some claim p is *false*, i.e., you want to show $\neg p$. The most promising strategy is to (a) suppose that p is true, and (b) show that this assumption leads to something you know to be false. "Something known to be false" may be a result that conflicts with—contradicts—p itself, or some other hypothesis of the overall claim you are considering. For instance, a proof of "if q then $\neg p$" involves assuming that q is true and proving $\neg p$ from it. If you prove $\neg p$ by contradiction, that means assuming p in addition to q. If you can prove $\neg q$ from p, you have shown that the assumption p leads to something that contradicts your other assumption q, since q and $\neg q$ cannot both be true. Of course, you have to use other inference patterns in your proof of the contradiction, as well as unpacking definitions. Let's consider an example.

Proposition prf.1. *If $X \subseteq Y$ and $Y = \emptyset$, then X has no elements.*

Proof. Suppose $X \subseteq Y$ and $Y = \emptyset$. We want to show that X has no elements.

> Since this is a conditional claim, we assume the antecedent and want to prove the consequent. The consequent is: X has no elements. We can make that a bit more explicit: it's not the case that there is an $x \in X$.

X has no elements iff it's not the case that there is an x such that $x \in X$.

> So we've determined that what we want to prove is really a negative claim $\neg p$, namely: it's not the case that there is an $x \in X$. To use proof by contradiction, we have to assume the corresponding positive claim p, i.e., there is an $x \in X$, and prove a contradiction from it. We indicate that we're doing a proof by contradiction by writing "by way of contradiction, assume" or even just "suppose not," and then state the assumption p.

Suppose not: there is an $x \in X$.

> This is now the new assumption we'll use to obtain a contradiction. We have two more assumptions: that $X \subseteq Y$ and that $Y = \emptyset$. The first gives us that $x \in Y$:

Since $X \subseteq Y$, $x \in Y$.

> But since $Y = \emptyset$, every element of Y (e.g., x) must also be an element of $\emptyset$.

Since $Y = \emptyset$, $x \in \emptyset$. This is a contradiction, since by definition $\emptyset$ has no elements.

> This already completes the proof: we've arrived at what we need (a contradiction) from the assumptions we've set up, and this means that the assumptions can't all be true. Since the first two assumptions ($X \subseteq Y$ and $Y = \emptyset$) are not contested, it must be the last assumption introduced (there is an $x \in X$) that must be false. But if we want to be thorough, we can spell this out.

Thus, our assumption that there is an $x \in X$ must be false, hence, X has no elements by proof by contradiction. □

Every positive claim is trivially equivalent to a negative claim: p iff $\neg\neg p$. So proofs by contradiction can also be used to establish positive claims "indirectly," as follows: To prove p, read it as the negative claim $\neg\neg p$. If we can prove a contradiction from $\neg p$, we've established $\neg\neg p$ by proof by contradiction, and hence p.

In the last example, we aimed to prove a negative claim, namely that X has no elements, and so the assumption we made for the purpose of proof by contradiction (i.e., that there is an $x \in X$) was a positive claim. It gave us something to work with, namely the hypothetical $x \in X$ about which we continued to reason until we got to $x \in \emptyset$.

When proving a positive claim indirectly, the assumption you'd make for the purpose of proof by contradiction would be negative. But very often you can easily reformulate a positive claim as a negative claim, and a negative claim as a positive claim. Our previous proof would have been essentially the same had we proved "$X = \emptyset$" instead of the negative consequent "X has no elements." (By definition of $=$, "$X = \emptyset$" is a general claim, since it unpacks to "every element of X is an element of $\emptyset$ and vice versa".) But it is easily seen to be equivalent to the negative claim "not: there is an $x \in X$."

So it is sometimes easier to work with $\neg p$ as an assumption than it is to prove p directly. Even when a direct proof is just as simple or even simpler (as in the next example), some people prefer to proceed indirectly. If the double negation confuses you, think of a proof by contradiction of some claim as a proof of a contradiction from the *opposite* claim. So, a proof by contradiction of $\neg p$ is a proof of a contradiction from the assumption p; and proof by contradiction of p is a proof of a contradiction from $\neg p$.

Proposition prf.2. $X \subseteq X \cup Y$.

Proof. We want to show that $X \subseteq X \cup Y$.

> On the face of it, this is a positive claim: every $x \in X$ is also in $X \cup Y$. The negation of that is: some $x \in X$ is $\notin X \cup Y$. So we can prove the claim indirectly by assuming this negated claim, and showing that it leads to a contradiction.

Suppose not, i.e., $X \nsubseteq X \cup Y$.

> We have a definition of $X \subseteq X \cup Y$: every $x \in X$ is also $\in X \cup Y$. To understand what $X \nsubseteq X \cup Y$ means, we have to use some elementary logical manipulation on the unpacked definition: it's false that every $x \in X$ is also $\in X \cup Y$ iff there is *some* $x \in X$ that is $\notin Z$. (This is a place where you want to be very careful: many students' attempted proofs by contradiction fail because they analyze the negation of a claim like "all As are Bs" incorrectly.) In other words, $X \nsubseteq X \cup Y$ iff there is an x such that $x \in X$ and $x \notin X \cup Y$. From then on, it's easy.

So, there is an $x \in X$ such that $x \notin X \cup Y$. By definition of $\cup$, $x \in X \cup Y$ iff $x \in X$ or $x \in Y$. Since $x \in X$, we have $x \in X \cup Y$. This contradicts the assumption that $x \notin X \cup Y$. □

Problem prf.1. Prove *indirectly* that $X \cap Y \subseteq X$.

Proposition prf.3. *If $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$.*

Proof. Suppose $X \subseteq Y$ and $Y \subseteq Z$. We want to show $X \subseteq Z$.

> Let's proceed indirectly: we assume the negation of what we want to etablish.

Suppose not, i.e., $X \nsubseteq Z$.

> As before, we reason that $X \nsubseteq Z$ iff not every $x \in X$ is also $\in Z$, i.e., some $x \in X$ is $\notin Z$. Don't worry, with practice you won't have to think hard anymore to unpack negations like this.

In other words, there is an x such that $x \in X$ and $x \notin Z$.

> Now we can use this to get to our contradiction. Of course, we'll have to use the other two assumptions to do it.

Since $X \subseteq Y$, $x \in Y$. Since $Y \subseteq Z$, $x \in Z$. But this contradicts $x \notin Z$. □

Proposition prf.4. *If $X \cup Y = X \cap Y$ then $X = Y$.*

Proof. Suppose $X \cup Y = X \cap Y$. We want to show that $X = Y$.

> The beginning is now routine:

Assume, by way of contradiction, that $X \neq Y$.

> Our assumption for the proof by contradiction is that $X \neq Y$. Since $X = Y$ iff $X \subseteq Y$ an $Y \subseteq X$, we get that $X \neq Y$ iff $X \nsubseteq Y$ *or* $Y \nsubseteq X$. (Note how important it is to be careful when manipulating negations!) To prove a contradiction from this disjunction, we use a proof by cases and show that in each case, a contradiction follows.

$X \neq Y$ iff $X \nsubseteq Y$ or $Y \nsubseteq X$. We distinguish cases.

> In the first case, we assume $X \nsubseteq Y$, i.e., for some x, $x \in X$ but $\notin Y$. $X \cap Y$ is defined as those elements that X and Y have in common, so if something isn't in one of them, it's not in the intersection. $X \cup Y$ is X together with Y, so anything in either is also in the union. This tells us that $x \in X \cup Y$ but $x \notin X \cap Y$, and hence that $X \cap Y \neq Y \cap X$.

Case 1: $X \nsubseteq Y$. Then for some x, $x \in X$ but $x \notin Y$. Since $x \notin Y$, then $x \notin X \cap Y$. Since $x \in X$, $x \in X \cup Y$. So, $X \cap Y \neq Y \cap X$, contradicting the assumption that $X \cap Y = X \cup Y$.

Case 2: $Y \nsubseteq X$. Then for some y, $y \in Y$ but $y \notin X$. As before, we have $y \in X \cup Y$ but $y \notin X \cap Y$, and so $X \cap Y \neq X \cup Y$, again contradicting $X \cap Y = X \cup Y$. □

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Bibliography