Proposition prf.1. If $A \subseteq C$, then $A \cup (C \setminus A) = C$.

Proof. Suppose that $A \subseteq C$. We want to show that $A \cup (C \setminus A) = C$.

We begin by observing that this is a conditional statement. It is tacitly universally quantified: the proposition holds for all sets $A$ and $C$. So $A$ and $C$ are variables for arbitrary sets. To prove such a statement, we assume the antecedent and prove the consequent.

We continue by using the assumption that $A \subseteq C$. Let’s unpack the definition of $\subseteq$: the assumption means that all elements of $A$ are also elements of $C$. Let’s write this down—it’s an important fact that we’ll use throughout the proof.

By the definition of $\subseteq$, since $A \subseteq C$, for all $z$, if $z \in A$, then $z \in C$.

We’ve unpacked all the definitions that are given to us in the assumption. Now we can move onto the conclusion. We want to show that $A \cup (C \setminus A) = C$, and so we set up a proof similarly to the last example: we show that every element of $A \cup (C \setminus A)$ is also an element of $C$ and, conversely, every element of $C$ is an element of $A \cup (C \setminus A)$. We can shorten this to: $A \cup (C \setminus A) \subseteq C$ and $C \subseteq (A \cup (C \setminus A))$. (Here we’re doing the opposite of unpacking a definition, but it makes the proof a bit easier to read.) Since this is a conjunction, we have to prove both parts. To show the first part, i.e., that every element of $(A \cup (C \setminus A))$ is also an element of $C$, we assume that $z \in (A \cup (C \setminus A))$ for an arbitrary $z$ and show that $z \in C$.

We’ve arrived at a disjunction, and from it we want to prove that $z \in C$. We do this using proof by cases.

Case 1: $z \in A$. Since for all $z$, if $z \in A$, $z \in C$, we have that $z \in C$.

Here we’ve used the fact recorded earlier which followed from the hypothesis of the proposition that $A \subseteq C$. The first case is complete, and we turn to the second case, $z \in (C \setminus A)$. Recall that $C \setminus A$ denotes the difference of the two sets, i.e., the set of all elements of $C$ which are not elements of $A$. But any element of $C$ not in $A$ is in particular an element of $C$.

Case 2: $z \in (C \setminus A)$. This means that $z \in C$ and $z \notin A$. So, in particular, $z \in C$. 

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Great, we’ve proved the first direction. Now for the second direction. Here we prove that $C \subseteq A \cup (C \setminus A)$. So we assume that $z \in C$ and prove that $z \in A \cup (C \setminus A)$.

Now let $z \in C$. We want to show that $z \in A$ or $z \in C \setminus A$.

Since all elements of $A$ are also elements of $C$, and $C \setminus A$ is the set of all things that are elements of $C$ but not $A$, it follows that $z$ is either in $A$ or in $C \setminus A$. This may be a bit unclear if you don’t already know why the result is true. It would be better to prove it step-by-step. It will help to use a simple fact which we can state without proof: $z \in A$ or $z \notin A$. This is called the “principle of excluded middle:” for any statement $p$, either $p$ is true or its negation is true. (Here, $p$ is the statement that $z \in A$.) Since this is a disjunction, we can again use proof-by-cases.

Either $z \in A$ or $z \notin A$. In the former case, $z \in A \cup (C \setminus A)$. In the latter case, $z \in C$ and $z \notin A$, so $z \in C \setminus A$. But then $z \in A \cup (C \setminus A)$.

Our proof is complete: we have shown that $A \cup (C \setminus A) = C$. $\Box$

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Bibliography