prf.1 Another Example

mth:prf:ex2:sec

Proposition prf.1. *If $X \subseteq Z$, then $X \cup (Z \setminus X) = Z$.*

Proof. Suppose that $X \subseteq Z$. We want to show that $X \cup (Z \setminus X) = Z$.

> We begin by observing that this is a conditional statement. It is tacitly universally quantified: the proposition holds for all sets X and Z. So X and Z are variables for arbitrary sets. To prove such a statement, we assume the antecedent and prove the consequent.
>
> We continue by using the assumption that $X \subseteq Z$. Let's unpack the definition of $\subseteq$: the assumption means that all elements of X are also elements of Z. Let's write this down—it's an important fact that we'll use throughout the proof.

By the definition of $\subseteq$, since $X \subseteq Z$, for all z, if $z \in X$, then $z \in Z$.

> We've unpacked all the definitions that are given to us in the assumption. Now we can move onto the conclusion. We want to show that $X \cup (Z \setminus X) = Z$, and so we set up a proof similarly to the last example: we show that every element of $X \cup (Z \setminus X)$ is also an element of Z and, conversely, every element of Z is an element of $X \cup (Z \setminus X)$. We can shorten this to: $X \cup (Z \setminus X) \subseteq Z$ and $Z \subseteq X \cup (Z \setminus X)$. (Here we're doing the opposite of unpacking a definition, but it makes the proof a bit easier to read.) Since this is a conjunction, we have to prove both parts. To show the first part, i.e., that every element of $X \cup (Z \setminus X)$ is also an element of Z, we assume that $z \in X \cup (Z \setminus X)$ for an arbitrary z and show that $z \in Z$. By the definition of $\cup$, we can conclude that $z \in X$ or $z \in Z \setminus X$ from $z \in X \cup (Z \setminus X)$. You should now be getting the hang of this.

$X \cup (Z \setminus X) = Z$ iff $X \cup (Z \setminus X) \subseteq Z$ and $Z \subseteq (X \cup (Z \setminus X)$. First we prove that $X \cup (Z \setminus X) \subseteq Z$. Let $z \in X \cup (Z \setminus X)$. So, either $z \in X$ or $z \in (Z \setminus X)$.

> We've arrived at a disjunction, and from it we want to prove that $z \in Z$. We do this using proof by cases.

Case 1: $z \in X$. Since for all z, if $z \in X$, $z \in Z$, we have that $z \in Z$.

> Here we've used the fact recorded earlier which followed from the hypothesis of the proposition that $X \subseteq Z$. The first case is complete, and we turn to the second case, $z \in (Z \setminus X)$. Recall that $Z \setminus X$ denotes the *difference* of the two sets, i.e., the set of all elements of Z which are not elements of X. But any element of Z not in X is in particular an element of Z.

Case 2: $z \in (Z \setminus X)$. This means that $z \in Z$ and $z \notin X$. So, in particular, $z \in Z$.

Great, we've proved the first direction. Now for the second direction. Here we prove that $Z \subseteq X \cup (Z \setminus X)$. So we assume that $z \in Z$ and prove that $z \in X \cup (Z \setminus X)$.

Now let $z \in Z$. We want to show that $z \in X$ or $z \in Z \setminus X$.

Since all elements of X are also elements of Z, and $Z \setminus X$ is the set of all things that are elements of Z but not X, it follows that z is either in X or in $Z \setminus X$. This may be a bit unclear if you don't already know why the result is true. It would be better to prove it step-by-step. It will help to use a simple fact which we can state without proof: $z \in X$ or $z \notin X$. This is called the "principle of excluded middle:" for any statement p, either p is true or its negation is true. (Here, p is the statement that $z \in X$.) Since this is a disjunction, we can again use proof-by-cases.

Either $z \in X$ or $z \notin X$. In the former case, $z \in X \cup (Z \setminus X)$. In the latter case, $z \in Z$ and $z \notin X$, so $z \in Z \setminus X$. But then $z \in X \cup (Z \setminus X)$.

Our proof is complete: we have shown that $X \cup (Z \setminus X) = Z$. $\square$

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Bibliography