ind.1 Structural Induction

mth:ind:sti:sec So far we have used induction to establish results about all natural numbers. But a corresponding principle can be used directly to prove results about all elements of an inductively defined set. This often called *structural* induction, because it depends on the structure of the inductively defined objects.

Generally, an inductive definition is given by (a) a list of "initial" elements of the set and (b) a list of operations which produce new elements of the set from old ones. In the case of parexpressions, for instance, the initial object is $\emptyset$ and the operations are

$$\begin{aligned} o_1(p) =& (p) \\ o_2(q, q') =& qq' \end{aligned}$$

You can even think of the natural numbers $\mathbb{N}$ themselves as being given be an inductive definition: the initial object is 0, and the operation is the successor function $x + 1$.

In order to prove something about all elements of an inductively defined set, i.e., that every element of the set has a property P, we must:

1. Prove that the initial objects have P
2. Prove that for each operation o, if the arguments have P, so does the result.

For instance, in order to prove something about all parexpressions, we would prove that it is true about $\emptyset$, that it is true of (p) provided it is true of p, and that it is true about qq' provided it is true of q and q' individually.

Proposition ind.1. *The number of* (*equals the number of*) *in any parexpression p.*

Proof. We use structural induction. Parexpressions are inductively defined, with initial object $\emptyset$ and the operations o_1 and o_2.

1. The claim is true for $\emptyset$, since the number of (in $\emptyset = 0$ and the number of) in $\emptyset$ also $= 0$.
2. Suppose the number of (in p equals the number of) in p. We have to show that this is also true for (p), i.e., $o_1(p)$. But the number of (in (p) is $1+$ the number of (in p. And the number of) in (p) is $1+$ the number of) in p, so the claim also holds for (p).
3. Suppose the number of (in q equals the number of), and the same is true for q'. The number of (in $o_2(p, p')$, i.e., in pp', is the sum of the number (in p and p'. The number of) in $o_2(p, p')$, i.e., in pp', is the sum of the number of) in p and p'. The number of (in $o_2(p, p')$ equals the number of) in $o_2(p, p')$.

The result follows by structural induction. □

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Bibliography