

ind.1 Structural Induction

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So far we have used induction to establish results about all natural numbers. But a corresponding principle can be used directly to prove results about all **elements** of an inductively defined set. This is often called *structural* induction, because it depends on the structure of the inductively defined objects.

Generally, an inductive definition is given by (a) a list of “initial” **elements** of the set and (b) a list of operations which produce new **elements** of the set from old ones. In the case of parexpressions, for instance, the initial object is \emptyset and the operations are

$$\begin{aligned}o_1(p) &= (p) \\ o_2(q, q') &= qq'\end{aligned}$$

You can even think of the natural numbers \mathbb{N} themselves as being given by an inductive definition: the initial object is 0, and the operation is the successor function $x + 1$.

In order to prove something about all elements of an inductively defined set, i.e., that every **element** of the set has a property P , we must:

1. Prove that the initial objects have P
2. Prove that for each operation o , if the arguments have P , so does the result.

For instance, in order to prove something about all parexpressions, we would prove that it is true about \emptyset , that it is true of (p) provided it is true of p , and that it is true about qq' provided it is true of q and q' individually.

Proposition ind.1. *The number of (equals the number of) in any parexpression p .*

Proof. We use structural induction. Parexpressions are inductively defined, with initial object \emptyset and the operations o_1 and o_2 .

1. The claim is true for \emptyset , since the number of (in $\emptyset = 0$ and the number of) in \emptyset also = 0.
2. Suppose the number of (in p equals the number of) in p . We have to show that this is also true for (p) , i.e., $o_1(p)$. But the number of (in (p) is $1 +$ the number of (in p . And the number of) in (p) is $1 +$ the number of) in p , so the claim also holds for (p) .
3. Suppose the number of (in q equals the number of), and the same is true for q' . The number of (in $o_2(p, q')$, i.e., in pp' , is the sum of the number (in p and q' . The number of) in $o_2(p, q')$, i.e., in pp' , is the sum of the number of) in p and q' . The number of (in $o_2(p, q')$ equals the number of) in $o_2(p, q')$.

The result follows by structural induction.

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Bibliography