

## ind.1 Strong Induction

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sec

In the principle of induction discussed above, we prove  $P(0)$  and also if  $P(n)$ , then  $P(n+1)$ . In the second part, we assume that  $P(n)$  is true and use this assumption to prove  $P(n+1)$ . Equivalently, of course, we could assume  $P(n-1)$  and use it to prove  $P(n)$ —the important part is that we be able to carry out the inference from any number to its successor; that we can prove the claim in question for any number under the assumption it holds for its predecessor.

There is a variant of the principle of induction in which we don't just assume that the claim holds for the predecessor  $n-1$  of  $n$ , but for all numbers smaller than  $n$ , and use this assumption to establish the claim for  $n$ . This also gives us the claim  $P(k)$  for all  $k \in \mathbb{N}$ . For once we have established  $P(0)$ , we have thereby established that  $P$  holds for all numbers less than 1. And if we know that if  $P(l)$  for all  $l < n$  then  $P(n)$ , we know this in particular for  $n = 1$ . So we can conclude  $P(2)$ . With this we have proved  $P(0)$ ,  $P(1)$ ,  $P(2)$ , i.e.,  $P(l)$  for all  $l < 3$ , and since we have also the conditional, if  $P(l)$  for all  $l < 3$ , then  $P(3)$ , we can conclude  $P(3)$ , and so on.

In fact, if we can establish the general conditional “for all  $n$ , if  $P(l)$  for all  $l < n$ , then  $P(n)$ ,” we do not have to establish  $P(0)$  anymore, since it follows from it. For remember that a general claim like “for all  $l < n$ ,  $P(l)$ ” is true if there are no  $l < n$ . This is a case of vacuous quantification: “all  $As$  are  $Bs$ ” is true if there are no  $As$ ,  $\forall x (\varphi(x) \rightarrow \psi(x))$  is true if no  $x$  satisfies  $\varphi(x)$ . In this case, the formalized version would be “ $\forall l (l < n \rightarrow P(l))$ ”—and that is true if there are no  $l < n$ . And if  $n = 0$  that's exactly the case: no  $l < 0$ , hence “for all  $l < 0$ ,  $P(l)$ ” is true, whatever  $P$  is. A proof of “if  $P(l)$  for all  $l < n$ , then  $P(n)$ ” thus automatically establishes  $P(0)$ .

This variant is useful if establishing the claim for  $n$  can't be made to just rely on the claim for  $n-1$  but may require the assumption that it is true for one or more  $l < n$ .

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## Bibliography