

## ind.1 Relations and Functions

mth:ind:rel:  
sec

When we have defined a set of objects (such as the natural numbers or the nice terms) inductively, we can also define *relations on* these objects by induction. For instance, consider the following idea: a nice term  $t$  is a subterm of a nice term  $t'$  if it occurs as a part of it. Let's use a symbol for it:  $t \sqsubseteq t'$ . Every nice term is a subterm of itself, of course:  $t \sqsubseteq t$ . We can give an inductive definition of this relation as follows:

**Definition ind.1.** The relation of a nice term  $t$  being a subterm of  $t'$ ,  $t \sqsubseteq t'$ , is defined by induction on  $s'$  as follows:

1. If  $t'$  is a letter, then  $t \sqsubseteq t'$  iff  $t = t'$ .
2. If  $t'$  is  $[s \circ s']$ , then  $t \sqsubseteq t'$  iff  $t = t'$ ,  $t \sqsubseteq s$ , or  $t \sqsubseteq s'$ .

This definition, for instance, will tell us that  $a \sqsubseteq [b \circ a]$ . For (2) says that  $a \sqsubseteq [b \circ a]$  iff  $a = [b \circ a]$ , or  $a \sqsubseteq b$ , or  $a \sqsubseteq a$ . The first two are false:  $a$  clearly isn't identical to  $[b \circ a]$ , and by (1),  $a \sqsubseteq b$  iff  $a = b$ , which is also false. However, also by (1),  $a \sqsubseteq a$  iff  $a = a$ , which is true.

It's important to note that the success of this definition depends on a fact that we haven't proved yet: every nice term  $t$  is either a letter by itself, or there are uniquely determined nice terms  $s$  and  $s'$  such that  $t = [s \circ s']$ . "Uniquely determined" here means that if  $t = [s \circ s']$  it isn't *also*  $= [r \circ r']$  with  $s \neq r$  or  $s' \neq r'$ . If this were the case, then clause (2) may come in conflict with itself: reading  $t'$  as  $[s \circ s']$  we might get  $t \sqsubseteq t'$ , but if we read  $t'$  as  $[r \circ r']$  we might get not  $t \sqsubseteq t'$ . Before we prove that this can't happen, let's look at an example where it *can* happen.

**Definition ind.2.** Define *bracketless terms* inductively by

1. Every letter is a bracketless term.
2. If  $s$  and  $s'$  are bracketless terms, then  $s \circ s'$  is a bracketless term.
3. Nothing else is a bracketless term.

Bracketless terms are, e.g.,  $a$ ,  $b \circ d$ ,  $b \circ a \circ b$ . Now if we defined "subterm" for bracketless terms the way we did above, the second clause would read

$$\text{If } t' = s \circ s', \text{ then } t \sqsubseteq t' \text{ iff } t = t', t \sqsubseteq s, \text{ or } t \sqsubseteq s'.$$

Now  $b \circ a \circ b$  is of the form  $s \circ s'$  with  $s = b$  and  $s' = a \circ b$ . It is also of the form  $r \circ r'$  with  $r = b \circ a$  and  $r' = b$ . Now is  $a \circ b$  a subterm of  $b \circ a \circ b$ ? The answer is yes if we go by the first reading, and no if we go by the second.

The property that the way a nice term is built up from other nice terms is unique is called *unique readability*. Since inductive definitions of relations for such inductively defined objects are important, we have to prove that it holds.

**Proposition ind.3.** *Suppose  $t$  is a nice term. Then either  $t$  is a letter by itself, or there are uniquely determined nice terms  $s$ ,  $s'$  such that  $t = [s \circ s']$ .*

*Proof.* If  $t$  is a letter by itself, the condition is satisfied. So assume  $t$  isn't a letter by itself. We can tell from the inductive definition that then  $t$  must be of the form  $[s \circ s']$  for some nice terms  $s$  and  $s'$ . It remains to show that these are uniquely determined, i.e., if  $t = [r \circ r']$ , then  $s = r$  and  $s' = r'$ .

So suppose  $t = [s \circ s']$  and  $t = [r \circ r']$  for nice terms  $s, s', r, r'$ . We have to show that  $s = r$  and  $s' = r'$ . First,  $s$  and  $r$  must be identical, for otherwise one is a proper initial segment of the other. But by ??, that is impossible if  $s$  and  $r$  are both nice terms. But if  $s = r$ , then clearly also  $s' = r'$ .  $\square$

We can also define functions inductively: e.g., we can define the function  $f$  that maps any nice term to the maximum depth of nested  $[ \dots ]$  in it as follows:

**Definition ind.4.** The *depth* of a nice term,  $f(t)$ , is defined inductively as follows: mth:ind:rel:  
defn:depth

$$\begin{aligned} f(s) &= 0 \text{ if } s \text{ is a letter} \\ f([s \circ s']) &= \max(f(s), f(s')) + 1 \end{aligned}$$

For instance

$$\begin{aligned} f([a \circ b]) &= \max(f(a), f(b)) + 1 = \\ &= \max(0, 0) + 1 = 1, \text{ and} \\ f([[a \circ b] \circ c]) &= \max(f([a \circ b]), f(c)) + 1 = \\ &= \max(1, 0) + 1 = 2. \end{aligned}$$

Here, of course, we assume that  $s$  and  $s'$  are nice terms, and make use of the fact that every nice term is either a letter or of the form  $[s \circ s']$ . It is again important that it can be of this form in only one way. To see why, consider again the bracketless terms we defined earlier. The corresponding “definition” would be:

$$\begin{aligned} g(s) &= 0 \text{ if } s \text{ is a letter} \\ g(s \circ s') &= \max(g(s), g(s')) + 1 \end{aligned}$$

Now consider the bracketless term  $a \circ b \circ c \circ d$ . It can be read in more than one way, e.g., as  $s \circ s'$  with  $s = a$  and  $s' = b \circ c \circ d$ , or as  $r \circ r'$  with  $r = a \circ b$  and  $r' = c \circ d$ . Calculating  $g$  according to the first way of reading it would give

$$\begin{aligned} g(s \circ s') &= \max(g(a), g(b \circ c \circ d)) + 1 = \\ &= \max(0, 2) + 1 = 3 \end{aligned}$$

while according to the other reading we get

$$\begin{aligned} g(r \circ r') &= \max(g(a \circ b), g(c \circ d)) + 1 = \\ &= \max(1, 1) + 1 = 2 \end{aligned}$$

But a function must always yield a unique value; so our “definition” of  $g$  doesn't define a function at all.

**Problem ind.1.** Give an inductive definition of the function  $l$ , where  $l(t)$  is the number of symbols in the nice term  $t$ .

**Problem ind.2.** Prove by induction on nice terms  $t$  that  $f(t) < l(t)$  (where  $l(t)$  is the number of symbols in  $t$  and  $f(t)$  is the depth of  $t$  as defined in [Definition ind.4](#)).

## Photo Credits

## Bibliography