ind.1 Relations and Functions

mth:ind:rel:sec When we have defined a set of objects (such as the natural numbers or the nice terms) inductively, we can also define *relations on* these objects by induction. For instance, consider the following idea: a nice term t is a subterm of a nice term t' if it occurs as a part of it. Let's use a symbol for it: $t \sqsubseteq t'$. Every nice term is a subterm of itself, of course: $t \sqsubseteq t$. We can give an inductive definition of this relation as follows:

Definition ind.1. The relation of a nice term t being a subterm of t', $t \sqsubseteq t'$, is defined by induction on s' as follows:

1. If t' is a letter, then $t \sqsubseteq t'$ iff $t = t'$.
2. If t' is $[s \circ s']$, then $t \sqsubseteq t'$ iff $t = t'$, $t \sqsubseteq s$, or $t \sqsubseteq s'$.

This definition, for instance, will tell us that $\mathrm{a} \sqsubseteq [\mathrm{b} \circ \mathrm{a}]$. For (2) says that $\mathrm{a} \sqsubseteq [\mathrm{b} \circ \mathrm{a}]$ iff $\mathrm{a} = [\mathrm{b} \circ \mathrm{a}]$, or $\mathrm{a} \sqsubseteq b$, or $\mathrm{a} \sqsubseteq \mathrm{a}$. The first two are false: a clearly isn't identical to $[\mathrm{b} \circ \mathrm{a}]$, and by (1), $\mathrm{a} \sqsubseteq \mathrm{b}$ iff $\mathrm{a} = \mathrm{b}$, which is also false. However, also by (1), $\mathrm{a} \sqsubseteq \mathrm{a}$ iff $\mathrm{a} = \mathrm{a}$, which is true.

It's important to note that the success of this definition depends on a fact that we haven't proved yet: every nice term t is either a letter by itself, or there are uniquely determined nice terms s and s' such that $t = [s \circ s']$. "Uniquely determined" here means that if $t = [s \circ s']$ it isn't *also* $= [r \circ r']$ with $s \neq r$ or $s' \neq r'$. If this were the case, then clause (2) may come in conflict with itself: reading t' as $[s \circ s']$ we might get $t \sqsubseteq t'$, but if we read t' as $[r \circ r']$ we might get not $t \sqsubseteq t'$. Before we prove that this can't happen, let's look at an example where it *can* happen.

Definition ind.2. Define *bracketless terms* inductively by

1. Every letter is a bracketles term.
2. If s and s' are bracketless terms, then $s \circ s'$ is a bracketless term.
3. Nothing else is a bracketless term.

Bracketless terms are, e.g., a, $\mathrm{b} \circ \mathrm{d}$, $\mathrm{b} \circ \mathrm{a} \circ \mathrm{b}$. Now if we defined "subterm" for bracketless terms the way we did above, the second clause would read

$$\text{If } t' = s \circ s', \text{ then } t \sqsubseteq t' \text{ iff } t = t',\ t \sqsubseteq s, \text{ or } t \sqsubseteq s'.$$

Now $\mathrm{b} \circ \mathrm{a} \circ \mathrm{b}$ is of the form $s \circ s'$ with $s = \mathrm{b}$ and $s' = \mathrm{a} \circ \mathrm{b}$. It is also of the form $r \circ r'$ with $r = \mathrm{b} \circ \mathrm{a}$ and $r' = \mathrm{b}$. Now is $\mathrm{a} \circ \mathrm{b}$ a subterm of $\mathrm{b} \circ \mathrm{a} \circ \mathrm{b}$? The answer is yes if we go by the first reading, and no if we go by the second.

The property that the way a nice term is built up from other nice terms is unique is called *unique readability.* Since inductive definitions of relations for such inductively defined objects are important, we have to prove that it holds.

Proposition ind.3. *Suppose t is a nice term. Then either t is a letter by itself, or there are uniquely determined nice terms s, s' such that $t = [s \circ s']$.*

Proof. If t is a letter by itself, the condition issatisfied. So assume t isn't a letter by itself. We can tell from the inductive definition that then t must be of the form $[s \circ s']$ for some nice terms s and s'. It remains to show that these are uniquely determined, i.e., if $t = [r \circ r']$, then $s = r$ and $s' = r'$.

So suppose $t = [s \circ s']$ and $t = [r \circ r']$ for nice terms s, s', r, r'. We have to show that $s = r$ and $s' = r'$. First, s and r must be identical, for otherwise one is a proper initial segment of the other. But by **??**, that is impossible if s and r are both nice terms. But if $s = r$, then clearly also $s' = r'$. $\square$

We can also define functions inductively: e.g., we can define the function f that maps any nice term to the maximum depth of nested $[\dots]$ in it as follows:

Definition ind.4. The *depth* of a nice term, $f(t)$, is defined inductively as follows: mth:ind:rel:defn:depth

$$
\begin{aligned}
f(s) &= 0 \text{ if } s \text{ is a letter} \\
f([s \circ s'] &= \max(f(s), f(s')) + 1
\end{aligned}
$$

For instance

$$
\begin{aligned}
f([\mathrm{a} \circ \mathrm{b}]) &= \max(f(\mathrm{a}), f(\mathrm{b})) + 1 = \\
&= \max(0, 0) + 1 = 1, \text{ and} \\
f([[\mathrm{a} \circ \mathrm{b}] \circ \mathrm{c}]) &= \max(f([\mathrm{a} \circ \mathrm{b}]), f(\mathrm{c})) + 1 = \\
&= \max(1, 0) + 1 = 2.
\end{aligned}
$$

Here, of course, we assume that s an s' are nice terms, and make use of the fact that every nice term is either a letter or of the form $[s \circ s']$. It is again important that it can be of this form in only one way. To see why, consider again the bracketless terms we defined earlier. The corresponding "definition" would be:

$$
\begin{aligned}
g(s) &= 0 \text{ if } s \text{ is a letter} \\
g(s \circ s') &= \max(g(s), g(s')) + 1
\end{aligned}
$$

Now consider the bracketless term $\mathrm{a} \circ \mathrm{b} \circ \mathrm{c} \circ \mathrm{d}$. It can be read in more than one way, e.g., as $s \circ s'$ with $s = \mathrm{a}$ and $s' = \mathrm{b} \circ \mathrm{c} \circ \mathrm{d}$, or as $r \circ r'$ with $r = \mathrm{a} \circ b$ and $r' = \mathrm{c} \circ \mathrm{d}$. Calculating g according to the first way of reading it would give

$$
\begin{aligned}
g(s \circ s') &= \max(g(\mathrm{a}), g(\mathrm{b} \circ \mathrm{c} \circ \mathrm{d})) + 1 = \\
&= \max(0, 2) + 1 = 3
\end{aligned}
$$

while according to the other reading we get

$$
\begin{aligned}
g(r \circ r') &= \max(g(\mathrm{a} \circ \mathrm{b}), g(\mathrm{c} \circ \mathrm{d})) + 1 = \\
&= \max(1, 1) + 1 = 2
\end{aligned}
$$

But a function must always yield a unique value; so our "definition" of g doesn't define a function at all.

Problem ind.1. Give an inductive definition of the function l, where $l(t)$ is the number of symbols in the nice term t.

Problem ind.2. Prove by induction on nice terms t that $f(t) < l(t)$ (where $l(t)$ is the number of symbols in t and $f(t)$ is the depth of t as defined in Definition ind.4).

Photo Credits

Bibliography