When we have defined a set of objects (such as the natural numbers or the nice terms) inductively, we can also define relations on these objects by induction. For instance, consider the following idea: a nice term $t$ is a subterm of a nice term $t'$ if it occurs as a part of it. Let's use a symbol for it: $t \sqsubseteq t'$. Every nice term is a subterm of itself, of course: $t \sqsubseteq t$. We can give an inductive definition of this relation as follows:

**Definition ind.1.** The relation of a nice term $t$ being a subterm of $t'$, $t \sqsubseteq t'$, is defined by induction on $s'$ as follows:

1. If $t'$ is a letter, then $t \sqsubseteq t'$ iff $t = t'$.
2. If $t'$ is $[s \circ s']$, then $t \sqsubseteq t'$ iff $t = t'$, $t \sqsubseteq s$, or $t \sqsubseteq s'$.

This definition, for instance, will tell us that $a \sqsubseteq [b \circ a]$. For (2) says that $a \sqsubseteq [b \circ a]$ iff $a = [b \circ a]$, or $a \sqsubseteq b$, or $a \sqsubseteq a$. The first two are false: $a$ clearly isn’t identical to $[b \circ a]$, and by (1), $a \sqsubseteq b$ iff $a = b$, which is also false. However, also by (1), $a \sqsubseteq a$ iff $a = a$, which is true.

It’s important to note that the success of this definition depends on a fact that we haven’t proved yet: every nice term $t$ is either a letter by itself, or there are uniquely determined nice terms $s$ and $s'$ such that $t = [s \circ s']$. “Uniquely determined” here means that if $t = [s \circ s']$ it isn’t also $= [r \circ r']$ with $s \neq r$ or $s' \neq r'$. If this were the case, then clause (2) may come in conflict with itself: reading $t'$ as $[s \circ s']$ we might get $t \sqsubseteq t'$, but if we read $t'$ as $[r \circ r']$ we might get not $t \sqsubseteq t'$. Before we prove that this can’t happen, let’s look at an example where it can happen.

**Definition ind.2.** Define *bracketless terms* inductively by

1. Every letter is a bracketless term.
2. If $s$ and $s'$ are bracketless terms, then $s \circ s'$ is a bracketless term.
3. Nothing else is a bracketless term.

Bracketless terms are, e.g., $a$, $b \circ d$, $b \circ a \circ b$. Now if we defined “subterm” for bracketless terms the way we did above, the second clause would read

If $t' = s \circ s'$, then $t \sqsubseteq t'$ iff $t = t'$, $t \sqsubseteq s$, or $t \sqsubseteq s'$.

Now $b \circ a \circ b$ is of the form $s \circ s'$ with $s = b$ and $s' = a \circ b$. It is also of the form $r \circ r'$ with $r = b \circ a$ and $r' = b$. Now is $a \circ b$ a subterm of $b \circ a \circ b$? The answer is yes if we go by the first reading, and no if we go by the second.

The property that the way a nice term is built up from other nice terms is unique is called *unique readability*. Since inductive definitions of relations for such inductively defined objects are important, we have to prove that it holds.

**Proposition ind.3.** Suppose $t$ is a nice term. Then either $t$ is a letter by itself, or there are uniquely determined nice terms $s$, $s'$ such that $t = [s \circ s']$. 

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Proof. If \( t \) is a letter by itself, the condition is satisfied. So assume \( t \) isn’t a letter by itself. We can tell from the inductive definition that then \( t \) must be of the form \([s \circ s']\) for some nice terms \( s \) and \( s' \). It remains to show that these are uniquely determined, i.e., if \( t = [r \circ r'] \), then \( s = r \) and \( s' = r' \).

So suppose \( t = [s \circ s'] \) and \( t = [r \circ r'] \) for nice terms \( s, s', r, r' \). We have to show that \( s = r \) and \( s' = r' \). First, \( s \) and \( r \) must be identical, for otherwise one is a proper initial segment of the other. But by ??, that is impossible if \( s \) and \( r \) are both nice terms. But if \( s = r \), then clearly also \( s' = r' \).

We can also define functions inductively: e.g., we can define the function \( f \) that maps any nice term to the maximum depth of nested \([\ldots]\) in it as follows:

**Definition ind.4.** The depth of a nice term, \( f(t) \), is defined inductively as follows:

\[
\begin{align*}
    f(s) &= 0 \text{ if } s \text{ is a letter} \\
    f([s \circ s']) &= \max(f(s), f(s')) + 1
\end{align*}
\]

For instance

\[
\begin{align*}
    f([a \circ b]) &= \max(f(a), f(b)) + 1 = \\
    &= \max(0, 0) + 1 = 1, \text{ and} \\
    f([[a \circ b] \circ c]) &= \max(f([a \circ b]), f(c)) + 1 = \\
    &= \max(1, 0) + 1 = 2.
\end{align*}
\]

Here, of course, we assume that \( s \) and \( s' \) are nice terms, and make use of the fact that every nice term is either a letter or of the form \([s \circ s']\). It is again important that it can be of this form in only one way. To see why, consider again the bracketless terms we defined earlier. The corresponding “definition” would be:

\[
\begin{align*}
    g(s) &= 0 \text{ if } s \text{ is a letter} \\
    g(s \circ s') &= \max(g(s), g(s')) + 1
\end{align*}
\]

Now consider the bracketless term \( a \circ b \circ c \circ d \). It can be read in more than one way, e.g., as \( s \circ s' \) with \( s = a \) and \( s' = b \circ c \circ d \), or as \( r \circ r' \) with \( r = a \circ b \) and \( r' = c \circ d \). Calculating \( g \) according to the first way of reading it would give

\[
\begin{align*}
    g(s \circ s') &= \max(g(a), g(b \circ c \circ d)) + 1 = \\
    &= \max(0, 2) + 1 = 3
\end{align*}
\]

while according to the other reading we get

\[
\begin{align*}
    g(r \circ r') &= \max(g(a \circ b), g(c \circ d)) + 1 = \\
    &= \max(1, 1) + 1 = 2
\end{align*}
\]

But a function must always yield a unique value; so our “definition” of \( g \) doesn’t define a function at all.
Problem ind.1. Give an inductive definition of the function $l$, where $l(t)$ is the number of symbols in the nice term $t$.

Problem ind.2. Prove by induction on nice terms $t$ that $f(t) < l(t)$ (where $l(t)$ is the number of symbols in $t$ and $f(t)$ is the depth of $t$ as defined in Definition ind.4).

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Bibliography