Relations and Functions

When we have defined a set of objects (such as the natural numbers or the nice terms) inductively, we can also define relations on these objects by induction.

For instance, consider the following idea: a nice term \( t \) is a subterm of a nice term \( t' \) if it occurs as a part of it. Let’s use a symbol for it: \( t \subseteq t' \). Every nice term is a subterm of itself, of course: \( t \subseteq t \). We can give an inductive definition of this relation as follows:

**Definition ind.1.** The relation of a nice term \( t \) being a subterm of \( t' \), \( t \subseteq t' \), is defined by induction on \( s' \) as follows:

1. If \( t' \) is a letter, then \( t \subseteq t' \) iff \( t = t' \).
2. If \( t' \) is \([s \circ s']\), then \( t \subseteq t' \) iff \( t = t' \), \( t \subseteq s \), or \( t \subseteq s' \).

This definition, for instance, will tell us that \( a \subseteq [b \circ a] \). For (2) says that \( a \subseteq [b \circ a] \) iff \( a = [b \circ a] \), or \( a \subseteq b \), or \( a \subseteq a \). The first two are false: \( a \) clearly isn’t identical to \([b \circ a]\), and by (1), \( a \subseteq b \) iff \( a = b \), which is also false. However, also by (1), \( a \subseteq a \) iff \( a = a \), which is true.

It’s important to note that the success of this definition depends on a fact that we haven’t proved yet: every nice term \( t \) is either a letter by itself, or there are uniquely determined nice terms \( s \) and \( s' \) such that \( t = [s \circ s'] \). “Uniquely determined” here means that if \( t = [s \circ s'] \) it isn’t also \( = [r \circ r'] \) with \( s \neq r \) or \( s' \neq r' \). If this were the case, then clause (2) may come in conflict with itself: reading \( t' \) as \([s \circ s']\) we might get \( t \subseteq t' \), but if we read \( t' \) as \([r \circ r']\) we might get not \( t \subseteq t' \). Before we prove that this can’t happen, let’s look at an example where it can happen.

**Definition ind.2.** Define bracketless terms inductively by

1. Every letter is a bracketless term.
2. If \( s \) and \( s' \) are bracketless terms, then \( s \circ s' \) is a bracketless term.
3. Nothing else is a bracketless term.

Bracketless terms are, e.g., \( a \), \( b \circ d \), \( b \circ a \circ b \). Now if we defined “subterm” for bracketless terms the way we did above, the second clause would read

\[
\text{If } t' = s \circ s', \text{ then } t \subseteq t' \text{ iff } t = t', \ t \subseteq s, \text{ or } t \subseteq s'.
\]

Now \( b \circ a \circ b \) is of the form \( s \circ s' \) with \( s = b \) and \( s' = a \circ b \). It is also of the form \( r \circ r' \) with \( r = b \circ a \) and \( r' = b \). Now is \( a \circ b \) a subterm of \( b \circ a \circ b \)? The answer is yes if we go by the first reading, and no if we go by the second.

The property that the way a nice term is built up from other nice terms is unique is called **unique readability**. Since inductive definitions of relations for such inductively defined objects are important, we have to prove that it holds.
Proposition ind.3. Suppose \( t \) is a nice term. Then either \( t \) is a letter by itself, or there are uniquely determined nice terms \( s, s' \) such that \( t = [s \circ s'] \).

Proof. If \( t \) is a letter by itself, the condition is satisfied. So assume \( t \) isn’t a letter by itself. We can tell from the inductive definition that then \( t \) must be of the form \([s \circ s']\) for some nice terms \( s \) and \( s' \). It remains to show that these are uniquely determined, i.e., if \( t = [r \circ r'] \), then \( s = r \) and \( s' = r' \).

So suppose \( t = [s \circ s'] \) and \( t = [r \circ r'] \) for nice terms \( s, s', r, r' \). We have to show that \( s = r \) and \( s' = r' \). First, \( s \) and \( r \) must be identical, for otherwise one is a proper initial segment of the other. But by ??, that is impossible if \( s \) and \( r \) are both nice terms. But if \( s = r \), then clearly also \( s' = r' \). \( \square \)

We can also define functions inductively: e.g., we can define the function \( f \) that maps any nice term to the maximum depth of nested \([\ldots]\) in it as follows:

Definition ind.4. The \textit{depth} of a nice term, \( f(t) \), is defined inductively as follows:

\[
\begin{align*}
    f(s) &= 0 \text{ if } s \text{ is a letter} \\
    f([s \circ s']) &= \max(f(s), f(s')) + 1
\end{align*}
\]

For instance

\[
\begin{align*}
    f([a \circ b]) &= \max(f(a), f(b)) + 1 = \\
    &= \max(0, 0) + 1 = 1, \text{ and} \\
    f([a \circ b] \circ c) &= \max(f([a \circ b]), f(c)) + 1 = \\
    &= \max(1, 0) + 1 = 2.
\end{align*}
\]

Here, of course, we assume that \( s \) and \( s' \) are nice terms, and make use of the fact that every nice term is either a letter or of the form \([s \circ s']\). It is again important that it can be of this form in only one way. To see why, consider again the bracketless terms we defined earlier. The corresponding “definition” would be:

\[
\begin{align*}
    g(s) &= 0 \text{ if } s \text{ is a letter} \\
    g(s \circ s') &= \max(g(s), g(s')) + 1
\end{align*}
\]

Now consider the bracketless term \( a \circ b \circ c \circ d \). It can be read in more than one way, e.g., as \( s \circ s' \) with \( s = a \) and \( s' = b \circ c \circ d \), or as \( r \circ r' \) with \( r = a \circ b \) and \( r' = c \circ d \). Calculating \( g \) according to the first way of reading it would give

\[
\begin{align*}
    g(s \circ s') &= \max(g(a), g(b \circ c \circ d)) + 1 = \\
    &= \max(0, 2) + 1 = 3
\end{align*}
\]

while according to the other reading we get

\[
\begin{align*}
    g(r \circ r') &= \max(g(a \circ b), g(c \circ d)) + 1 = \\
    &= \max(1, 1) + 1 = 2
\end{align*}
\]
But a function must always yield a unique value; so our “definition” of $g$ doesn’t define a function at all.

**Problem ind.1.** Give an inductive definition of the function $l$, where $l(t)$ is the number of symbols in the nice term $t$.

**Problem ind.2.** Prove by induction on nice terms $t$ that $f(t) < l(t)$ (where $l(t)$ is the number of symbols in $t$ and $f(t)$ is the depth of $t$ as defined in Definition ind.4).

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Bibliography