Relations and Functions

When we have defined a set of objects (such as the natural numbers or the nice terms) inductively, we can also define relations on these objects by induction. For instance, consider the following idea: a nice term \( t \) is a subterm of a nice term \( t' \) if it occurs as a part of it. Let’s use a symbol for it: \( t \sqsubseteq t' \). Every nice term is a subterm of itself, of course: \( t \sqsubseteq t \). We can give an inductive definition of this relation as follows:

**Definition ind.1.** The relation of a nice term \( t \) being a subterm of \( t' \), \( t \sqsubseteq t' \), is defined by induction on \( s' \) as follows:

1. If \( t' \) is a letter, then \( t \sqsubseteq t' \) iff \( t = t' \).
2. If \( t' \) is \( [s \circ s'] \), then \( t \sqsubseteq t' \) iff \( t = t' \), \( t \sqsubseteq s \), or \( t \sqsubseteq s' \).

This definition, for instance, will tell us that \( a \sqsubseteq [b \circ a] \). For (2) says that \( a \sqsubseteq [b \circ a] \) iff \( a = [b \circ a] \), or \( a \sqsubseteq b \), or \( a \sqsubseteq a \). The first two are false: \( a \) clearly isn’t identical to \( [b \circ a] \), and by (1), \( a \sqsubseteq b \) iff \( a = b \), which is also false. However, also by (1), \( a \sqsubseteq a \) iff \( a = a \), which is true.

It’s important to note that the success of this definition depends on a fact that we haven’t proved yet: every nice term \( t \) is either a letter by itself, or there are uniquely determined nice terms \( s \) and \( s' \) such that \( t = [s \circ s'] \). “Uniquely determined” here means that if \( t = [s \circ s'] \) it isn’t also \( = [r \circ r'] \) with \( s \neq r \) or \( s' \neq r' \). If this were the case, then clause (2) may come in conflict with itself: reading \( t' \) as \( [s \circ s'] \) we might get \( t \sqsubseteq t' \), but if we read \( t' \) as \( [r \circ r'] \) we might get not \( t \sqsubseteq t' \). Before we prove that this can’t happen, let’s look at an example where it can happen.

**Definition ind.2.** Define bracketless terms inductively by

1. Every letter is a bracketless term.
2. If \( s \) and \( s' \) are bracketless terms, then \( s \circ s' \) is a bracketless term.
3. Nothing else is a bracketless term.

Bracketless terms are, e.g., \( a, b \circ d, b \circ a \circ b \). Now if we defined “subterm” for bracketless terms the way we did above, the second clause would read

If \( t' = s \circ s' \), then \( t \sqsubseteq t' \) iff \( t = t' \), \( t \sqsubseteq s \), or \( t \sqsubseteq s' \).

Now \( b \circ a \circ b \) is of the form \( s \circ s' \) with \( s = b \) and \( s' = a \circ b \). It is also of the form \( r \circ r' \) with \( r = b \circ a \) and \( r' = b \). Now is \( a \circ b \) a subterm of \( b \circ a \circ b \)? The answer is yes if we go by the first reading, and no if we go by the second.

The property that the way a nice term is built up from other nice terms is unique is called unique readability. Since inductive definitions of relations for such inductively defined objects are important, we have to prove that it holds.

**Proposition ind.3.** Suppose \( t \) is a nice term. Then either \( t \) is a letter by itself, or there are uniquely determined nice terms \( s, s' \) such that \( t = [s \circ s'] \).
Proof. If $t$ is a letter by itself, the condition is satisfied. So assume $t$ isn’t a letter by itself. We can tell from the inductive definition that then $t$ must be of the form $[s \circ s']$ for some nice terms $s$ and $s'$. It remains to show that these are uniquely determined, i.e., if $t = [r \circ r']$, then $s = r$ and $s' = r'$.

So suppose $t = [s \circ s']$ and $t = [r \circ r']$ for nice terms $s$, $s'$, $r$, $r'$. We have to show that $s = r$ and $s' = r'$. First, $s$ and $r$ must be identical, for otherwise one is a proper initial segment of the other. But by ??, that is impossible if $s$ and $r$ are both nice terms. But if $s = r$, then clearly also $s' = r'$.

We can also define functions inductively: e.g., we can define the function $f$ that maps any nice term to the maximum depth of nested $[\ldots]$ in it as follows:

**Definition ind.4.** The depth of a nice term, $f(t)$, is defined inductively as follows:

\[
\begin{align*}
f(s) &= 0 \text{ if } s \text{ is a letter} \\
f([s \circ s']) &= \max(f(s), f(s')) + 1
\end{align*}
\]

For instance

\[
\begin{align*}
f([a \circ b]) &= \max(f(a), f(b)) + 1 = \\
&= \max(0, 0) + 1 = 1, \text{ and} \\
f([a \circ b] \circ c) &= \max(f([a \circ b]), f(c)) + 1 = \\
&= \max(1, 0) + 1 = 2.
\end{align*}
\]

Here, of course, we assume that $s$ and $s'$ are nice terms, and make use of the fact that every nice term is either a letter or of the form $[s \circ s']$. It is again important that it can be of this form in only one way. To see why, consider again the bracketless terms we defined earlier. The corresponding “definition” would be:

\[
\begin{align*}
g(s) &= 0 \text{ if } s \text{ is a letter} \\
g(s \circ s') &= \max(g(s), g(s')) + 1
\end{align*}
\]

Now consider the bracketless term $a \circ b \circ c \circ d$. It can be read in more than one way, e.g., as $s \circ s'$ with $s = a$ and $s' = b \circ c \circ d$, or as $r \circ r'$ with $r = a \circ b$ and $r' = c \circ d$. Calculating $g$ according to the first way of reading it would give

\[
\begin{align*}
g(s \circ s') &= \max(g(a), g(b \circ c \circ d)) + 1 = \\
&= \max(0, 2) + 1 = 3
\end{align*}
\]

while according to the other reading we get

\[
\begin{align*}
g(r \circ r') &= \max(g(a \circ b), g(c \circ d)) + 1 = \\
&= \max(1, 1) + 1 = 2
\end{align*}
\]

But a function must always yield a unique value; so our “definition” of $g$ doesn’t define a function at all.
Problem ind.1. Give an inductive definition of the function $l$, where $l(t)$ is the number of symbols in the nice term $t$.

Problem ind.2. Prove by induction on nice terms $t$ that $f(t) < l(t)$ (where $l(t)$ is the number of symbols in $t$ and $f(t)$ is the depth of $t$ as defined in Definition ind.4).

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Bibliography