Relations and Functions

When we have defined a set of objects (such as the natural numbers or the nice terms) inductively, we can also define \textit{relations on} these objects by induction. For instance, consider the following idea: a nice term \( t_1 \) is a subterm of a nice term \( t_2 \) if it occurs as a part of it. Let’s use a symbol for it: \( t_1 \sqsubseteq t_2 \). Every nice term is a subterm of itself, of course: \( t \sqsubseteq t \). We can give an inductive definition of this relation as follows:

**Definition ind.1.** The relation of a nice term \( t_1 \) being a subterm of \( t_2 \), \( t_1 \sqsubseteq t_2 \), is defined by induction on \( t_2 \) as follows:

1. If \( t_2 \) is a letter, then \( t_1 \sqsubseteq t_2 \iff t_1 = t_2 \).
2. If \( t_2 = [s_1 \circ s_2] \), then \( t_1 \sqsubseteq t_2 \iff t = t_2, t_1 \sqsubseteq s_1 \), or \( t_1 \sqsubseteq s_2 \).

This definition, for instance, will tell us that \( a \sqsubseteq [b \circ a] \). For (2) says that \( a \sqsubseteq [b \circ a] \) if \( a = [b \circ a] \), or \( a \sqsubseteq b \), or \( a \sqsubseteq a \). The first two are false: \( a \) clearly isn’t identical to \([b \circ a] \), and by (1), \( a \sqsubseteq b \) iff \( a = b \), which is also false. However, also by (1), \( a \sqsubseteq a \) iff \( a = a \), which is true.

It’s important to note that the success of this definition depends on a fact that we haven’t proved yet: every nice term \( t \) is either a letter by itself, or there are \textit{uniquely determined} nice terms \( s_1 \) and \( s_2 \) such that \( t = [s_1 \circ s_2] \). “Uniquely determined” here means that if \( t = [s_1 \circ s_2] \) it isn’t also \( [r_1 \circ r_2] \) with \( s_1 \neq r_1 \) or \( s_2 \neq r_2 \). If this were the case, then clause (2) may come in conflict with itself: reading \( t_2 \) as \([s_1 \circ s_2] \) we might get \( t_1 \sqsubseteq t_2 \), but if we read \( t_2 \) as \([r_1 \circ r_2] \) we might get not \( t_1 \sqsubseteq t_2 \). Before we prove that this can’t happen, let’s look at an example where it can happen.

**Definition ind.2.** Define \textit{bracketless terms} inductively by

1. Every letter is a bracketless term.
2. If \( s_1 \) and \( s_2 \) are bracketless terms, then \( s_1 \circ s_2 \) is a bracketless term.
3. Nothing else is a bracketless term.

Bracketless terms are, e.g., \( a, b \circ d, b \circ a \circ b \). Now if we defined “subterm” for bracketless terms the way we did above, the second clause would read

If \( t_2 = s_1 \circ s_2 \), then \( t_1 \sqsubseteq t_2 \iff t_1 = t_2, t_1 \sqsubseteq s_1 \), or \( t_1 \sqsubseteq s_2 \).

Now \( b \circ a \circ b \) is of the form \( s_1 \circ s_2 \) with

\[
s_1 = b \quad \text{and} \quad s_2 = a \circ b.
\]

It is also of the form \( r_1 \circ r_2 \) with

\[
r_1 = b \circ a \quad \text{and} \quad r_2 = b.
\]
Now is $a \circ b$ a subterm of $b \circ a \circ b$? The answer is yes if we go by the first reading, and no if we go by the second.

The property that the way a nice term is built up from other nice terms is unique is called \textit{unique readability}. Since inductive definitions of relations for such inductively defined objects are important, we have to prove that it holds.

**Proposition ind.3.** Suppose $t$ is a nice term. Then either $t$ is a letter by itself, or there are uniquely determined nice terms $s_1, s_2$ such that $t = [s_1 \circ s_2]$.

**Proof.** If $t$ is a letter by itself, the condition is satisfied. So assume $t$ isn’t a letter by itself. We can tell from the inductive definition that then $t$ must be of the form $[s_1 \circ s_2]$ for some nice terms $s_1$ and $s_2$. It remains to show that these are uniquely determined, i.e., if $t = [r_1 \circ r_2]$, then $s_1 = r_1$ and $s_2 = r_2$.

So suppose $t = [s_1 \circ s_2]$ and also $t = [r_1 \circ r_2]$ for nice terms $s_1, s_2, r_1, r_2$. We have to show that $s_1 = r_1$ and $s_2 = r_2$. First, $s_1$ and $r_1$ must be identical, for otherwise one is a proper initial segment of the other. But by ??, that is impossible if $s_1$ and $r_1$ are both nice terms. But if $s_1 = r_1$, then clearly also $s_2 = r_2$. \qed

We can also define functions inductively: e.g., we can define the function $f$ that maps any nice term to the maximum depth of nested $[\ldots]$ in it as follows:

**Definition ind.4.** The \textit{depth} of a nice term, $f(t)$, is defined inductively as follows:

$$f(t) = \begin{cases} 0 & \text{if } t \text{ is a letter} \\ \max(f(s), f(s')) + 1 & \text{if } t = [s_1 \circ s_2]. \end{cases}$$

For instance

$$f([a \circ b]) = \max(f(a), f(b)) + 1 = \max(0, 0) + 1 = 1,$$

and

$$f([a \circ b] \circ c) = \max(f([a \circ b]), f(c)) + 1 = \max(1, 0) + 1 = 2.$$ 

Here, of course, we assume that $s_1$ an $s_2$ are nice terms, and make use of the fact that every nice term is either a letter or of the form $[s_1 \circ s_2]$. It is again important that it can be of this form in only one way. To see why, consider again the bracketless terms we defined earlier. The corresponding “definition” would be:

$$g(t) = \begin{cases} 0 & \text{if } t \text{ is a letter} \\ \max(g(s), g(s')) + 1 & \text{if } t = [s_1 \circ s_2]. \end{cases}$$

Now consider the bracketless term $a \circ b \circ c \circ d$. It can be read in more than one way, e.g., as $s_1 \circ s_2$ with

$s_1 = a$ and $s_2 = b \circ c \circ d$. 

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or as \( r_1 \circ r_2 \) with
\[
  r_1 = a \circ b \quad \text{and} \quad r_2 = c \circ d.
\]
Calculating \( g \) according to the first way of reading it would give
\[
g(s_1 \circ s_2) = \max(g(a), g(b \circ c \circ d)) + 1 = \\
= \max(0, 2) + 1 = 3
\]
while according to the other reading we get
\[
g(r_1 \circ r_2) = \max(g(a \circ b), g(c \circ d)) + 1 = \\
= \max(1, 1) + 1 = 2
\]
But a function must always yield a unique value; so our “definition” of \( g \) doesn’t define a function at all.

**Problem ind.1.** Give an inductive definition of the function \( l \), where \( l(t) \) is the number of symbols in the nice term \( t \).

**Problem ind.2.** Prove by structural induction on nice terms \( t \) that \( f(t) < l(t) \) (where \( l(t) \) is the number of symbols in \( t \) and \( f(t) \) is the depth of \( t \) as defined in Definition ind.4).