ind.1 Inductive Definitions

mth:ind:idf:sec In logic we very often define kinds of objects *inductively*, i.e., by specifying rules for what counts as an object of the kind to be defined which explain how to get new objects of that kind from old objects of that kind. For instance, we often define special kinds of sequences of symbols, such as the terms and formulas of a language, by induction. For a simple example, consider strings of consisting of letters a, b, c, d, the symbol $\circ$, and brackets [and], such as "[[c $\circ$ d][", "[a[]$\circ$]", "a" or "[[a $\circ$ b] $\circ$ d]". You probably feel that there's something "wrong" with the first two strings: the brackets don't "balance" at all in the first, and you might feel that the "$\circ$" should "connect" expressions that themselves make sense. The third and fourth string look better: for every "[" there's a closing "]" (if there are any at all), and for any $\circ$ we can find "nice" expressions on either side, surrounded by a pair of parenteses.

We would like to precisely specify what counts as a "nice term." First of all, every letter by itself is nice. Anything that's not just a letter by itself should be of the form "$[t \circ s]$" where s and t are themselves nice. Conversely, if t and s are nice, then we can form a new nice term by putting a $\circ$ between them and surround them by a pair of brackets. We might use these operations to *define* the set of nice terms. This is an *inductive definition*.

Definition ind.1 (Nice terms)**.** The set of *nice terms* is inductively defined as follows:

1. Any letter a, b, c, d is a nice term.
2. If s and s' are nice terms, then so is $[s \circ s']$.
3. Nothing else is a nice term.

This definition tells us that something counts as a nice term iff it can be constructed according to the two conditions (1) and (2) in some finite number of steps. In the first step, we construct all nice terms just consisting of letters by themselves, i.e.,

$$\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$$

In the second step, we apply (2) to the terms we've constructed. We'll get

$$[\mathrm{a} \circ \mathrm{a}], [\mathrm{a} \circ \mathrm{b}], [\mathrm{b} \circ \mathrm{a}], \dots, [\mathrm{d} \circ \mathrm{d}]$$

for all combinations of two letters. In the third step, we apply (2) again, to any two nice terms we've constructed so far. We get new nice term such as $[\mathrm{a} \circ [\mathrm{a} \circ \mathrm{a}]]$—where t is a from step 1 and s is $[\mathrm{a} \circ \mathrm{a}]$ from step 2—and $[[\mathrm{b} \circ \mathrm{c}] \circ [\mathrm{d} \circ \mathrm{b}]]$ constructed out of the two terms $[\mathrm{b} \circ \mathrm{c}]$ and $[\mathrm{d} \circ \mathrm{b}]$ from step 2. And so on. Clause (3) rules out that anything not constructed in this way sneaks into the set of nice terms.

Note that we have not yet proved that every sequence of symbols that "feels" nice is nice according to this definition. However, it should be clear that everything we can construct does in fact "feel nice:" brackets are balanced, and $\circ$ connects parts that are themselves nice.

The key feature of inductive definitions is that if you want to prove something about all nice terms, the definition tells you which cases you must consider. For instance, if you are told that t is a nice term, the inductive definition tells you what t can look like: t can be a letter, or it can be $[r \circ s]$ for some other pair of nice terms r and s. Because of clause (3), those are the only possibilities.

When proving claims about all of an inductively defined set, the strong form of induction becomes particularly important. For instance, suppose we want to prove that for every nice term of length n, the number of [in it is $< n/2$. This can be seen as a claim about all n: for every n, the number of [in any nice term of length n is $< n/2$.

Proposition ind.2. *For any n, the number of* [*in a nice term of length n is $< n/2$.*

Proof. To prove this result by (strong) induction, we have to show that the following conditional claim is true:

> If for every $k < n$, any parexpression of length k has $k/2$ ['s, then any parexpression of length n has $n/2$ ['s.

To show this conditional, assume that its antecedent is true, i.e., assume that for any $k < n$, parexpressions of length k contain $< k/2$ ['s. We call this assumption the inductive hypothesis. We want to show the same is true for parexpressions of length n.

So suppose t is a nice term of length n. Because parexpressions are inductively defined, we have three two cases: (1) t is a letter by itself, or t is $[r \circ s]$ for some nice terms r and s.

1. t is a letter. Then $n = 1$, and the number of [in t is 0. Since $0 < 1/2$, the claim holds.

2. t is $[s \circ s']$ for some nice terms s and s'. Let's let k be the length of s and k' be the length of s'. Then the length n of t is $k + k' + 3$ (the lengths of s and s' plus three symbols [, $\circ$,]). Since $k + k' + 3$ is always greater than k, $k < n$. Similarly, $k' < n$. That means that the induction hypothesis applies to the terms s and s': the number m of [in s is $< k/2$, and the number of [in s' is $< k'/2$.

   The number of [in t is the number of [in s, plus the number of [in s', plus 1, i.e., it is $m + m' + 1$. Since $m < k/2$ and $m' < k'/2$ we have:

$$m + m' + 1 < \frac{k}{2} + \frac{k'}{2} + 1 = \frac{k + k' + 2}{2} < \frac{k + k' + 3}{2} = n/2.$$

In each case, we've shown that the number of [in t is $< n/2$ (on the basis of the inductive hypothesis). By strong induction, the proposition follows. □

Problem ind.1. Define the set of supernice terms by

1. Any letter a, b, c, d is a supernice term.
2. If s is a supernice term, then so is $[s]$.
3. If t and s are supernice terms, then so is $[t \circ s]$.
4. Nothing else is a supernice term.

Show that the number of [in a supernice term s of length n is $\leq n/2 + 1$.

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Bibliography