thr.1 Designating not just $\mathbb{T}$

mvl:thr:mul:sec

So far the logics we've seen all had the set of designated truth values $V^+ = \{\mathbb{T}\}$, i.e., something counts as true iff its truth value is $\mathbb{T}$. But one might also count something as true if it's just not $\mathbb{F}$. Then one would get a logic by stipulating in the matrix, e.g., that $V^+ = \{\mathbb{T}, \mathbb{U}\}$.

Definition thr.1. The *logic of paradox* **LP** is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\wedge$, $\vee$, $\rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ and $\mathbb{U}$ are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as in strong Kleene logic.

Definition thr.2. Halldén's *logic of nonsense* **Hal** is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\wedge$, $\vee$, $\rightarrow$ and a 1-place connective $+$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ and $\mathbb{U}$ are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as weak Kleene logic, plus the "is meaningless" operator:

$\widetilde{+}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$

By contrast to the Kleene logics with which they share truth tables, these *do* have tautologies.

mvl:thr:mul:prop:LP-taut-CL

Proposition thr.3. *The tautologies of* **LP** *are the same as the tautologies of classical propositional logic.*

Proof. By **??**, if $\vDash_{\mathbf{LP}} \varphi$ then $\vDash_{\mathbf{C}} \varphi$. To show the reverse, we show that if there is a valuation $\mathfrak{v} \colon \mathrm{At}_0 \to \{\mathbb{F}, \mathbb{T}, \mathbb{U}\}$ such that $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$ then there is a valuation $\mathfrak{v}' \colon \mathrm{At}_0 \to \{\mathbb{F}, \mathbb{T}\}$ such that $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{F}$. This establishes the result for **LP**, since **Ks** and **LP** have the same characteristic truth functions, and $\mathbb{F}$ is the only truth value of **LP** that is not designated (that is the only difference between **LP** and **Ks**). Thus, if $\nvDash_{\mathbf{LP}} \varphi$, for some valuation $\mathfrak{v}$, $\overline{\mathfrak{v}}_{\mathbf{LP}}(\varphi) = \overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$. By the claim we're proving, $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{F}$, i.e., $\nvDash_{\mathbf{C}} \varphi$.

To establish the claim, we first define $\mathfrak{v}'$ as

$$\mathfrak{v}'(p) = \begin{cases} \mathbb{T} & \text{if } \mathfrak{v}(p) \in \{\mathbb{T}, \mathbb{U}\} \\ \mathbb{F} & \text{otherwise} \end{cases}$$

We now show by induction on φ that (a) if $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$ then $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{F}$, and (b) if $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{T}$ then $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{T}$

1. Induction basis: $\varphi \equiv p$. By **??**, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathfrak{v}(p) = \overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi)$, which implies both (a) and (b).

   For the induction step, consider the cases:

2. $\varphi \equiv \neg\psi$.

   a) Suppose $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\neg\psi) = \mathbb{F}$. By the definition of $\tilde{\neg}_{\mathbf{Ks}}$, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$. By inductive hypothesis, case (b), we get $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi) = \mathbb{T}$, so $\overline{\mathfrak{v}'}_{\mathbf{C}}(\neg\psi) = \mathbb{F}$.
   b) Suppose $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\neg\psi) = \mathbb{T}$. By the definition of $\tilde{\neg}_{\mathbf{Ks}}$, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$. By inductive hypothesis, case (a), we get $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi) = \mathbb{F}$, so $\overline{\mathfrak{v}'}_{\mathbf{C}}(\neg\psi) = \mathbb{T}$.

3. $\varphi \equiv (\psi \land \chi)$.

   a) Suppose $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi \land \chi) = \mathbb{F}$. By the definition of $\widetilde{\land}_{\mathbf{Ks}}$, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$ or $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$. By inductive hypothesis, case (a), we get $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi) = \mathbb{F}$ or $\overline{\mathfrak{v}'}_{\mathbf{C}}(\chi) = \mathbb{F}$, so $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi \land \chi) = \mathbb{F}$.
   b) Suppose $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi \land \chi) = \mathbb{T}$. By the definition of $\widetilde{\land}_{\mathbf{Ks}}$, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$ and $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$. By inductive hypothesis, case (b), we get $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi) = \mathbb{T}$ and $\overline{\mathfrak{v}'}_{\mathbf{C}}(\chi) = \mathbb{T}$, so $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi \land \chi) = \mathbb{T}$.

The other two cases are similar, and left as exercises. Alternatively, the proof above establishes the result for all formulas only containing $\neg$ and $\land$. One may now appeal to the facts that in both **Ks** and **C**, for any $\mathfrak{v}$, $\overline{\mathfrak{v}}(\psi \lor \chi) = \overline{\mathfrak{v}}(\neg(\neg\psi \land \neg\chi))$ and $\overline{\mathfrak{v}}(\psi \to \chi) = \overline{\mathfrak{v}}(\neg(\psi \land \neg\chi))$. □

Problem thr.1. Complete the proof Proposition thr.3, i.e., establish (a) and (b) for the cases where $\varphi \equiv (\psi \lor \chi)$ and $\varphi \equiv (\psi \to \chi)$.

Problem thr.2. Prove that every classical tautology is a tautology in **Hal**.

Although they have the same tautologies as classical logic, their consequence relations are different. **LP**, for instance, is *paraconsistent* in that $\neg p, p \not\vDash q$, and so the principle of explosion $\neg\varphi, \varphi \vDash \psi$ does not hold in general. (It holds for some cases of φ and ψ, e.g., if ψ is a tautology.)

Problem thr.3. Which of the following relations hold in (a) **LP** and in (b) **Hal**? Give a truth table for each.

1. $p, p \to q \vDash q$
2. $\neg q, p \to q \vDash \neg p$
3. $p \lor q, \neg p \vDash q$
4. $\neg p, p \vDash q$
5. $p \vDash p \lor q$

6. $p \to q, q \to r \vDash p \to r$

What if you make $\mathbb{U}$ designated in $\mathbf{Ł}_3$?

Definition thr.4. The logic *3-valued R-Mingle* $\mathbf{RM}_3$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot$, $\neg$, $\wedge$, $\vee$, $\to$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ and $\mathbb{U}$ are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as Łukasiewicz logic $\mathbf{Ł}_3$.

Problem thr.4. Which of the following relations hold in $\mathbf{RM}_3$?

1. $p, p \to q \vDash q$
2. $p \vee q, \neg p \vDash q$
3. $\neg p, p \vDash q$
4. $p \vDash p \vee q$

Different truth tables can sometimes generate the same logic (entailment relation) just by changing the designated values. E.g., this happens if in Gödel logic we take $V^+ = \{\mathbb{T}, \mathbb{U}\}$ instead of $\{\mathbb{T}\}$.

mvl:thr:mul:prop:gl-udes **Proposition thr.5.** *The matrix with $V = \{\mathbb{F}, \mathbb{U}, \mathbb{T}\}$, $V^+ = \{\mathbb{T}, \mathbb{U}\}$, and the truth functions of 3-valued Gödel logic defines classical logic.*

Proof. Exercise. □

Problem thr.5. Prove Proposition thr.5 by showing that for the logic $\mathbf{L}$ defined just like Gödel logic but with $V^+ = \{\mathbb{T}, \mathbb{U}\}$, if $\Gamma \nvDash_{\mathbf{L}} \psi$ then $\Gamma \nvDash_{\mathbf{C}} \psi$. Use the ideas of Proposition thr.3, except instead of proving properties (a) and (b), show that $\overline{\mathfrak{v}}_{\mathbf{G}}(\varphi) = \mathbb{F}$ iff $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{F}$ (and hence that $\overline{\mathfrak{v}}_{\mathbf{G}}(\varphi) \in \{\mathbb{T}, \mathbb{U}\}$ iff $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{T}$). Explain why this establishes the proposition.

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Bibliography