Designating not just \( T \)

So far the logics we’ve seen all had the set of designated truth values \( V^+ = \{ T \} \), i.e., something counts as true iff its truth value is \( T \). But one might also count something as true if it’s just not \( F \). Then one would get a logic by stipulating in the matrix, e.g., that \( V^+ = \{ T, U \} \).

**Definition thr.1.** The *logic of paradox* \( LP \) is defined using the matrix:

1. The standard propositional language \( L_0 \) with \( \neg, \wedge, \vee, \rightarrow \).
2. The set of truth values \( V = \{ T, U, F \} \).
3. \( T \) and \( U \) are designated, i.e., \( V^+ = \{ T, U \} \).
4. Truth functions are the same as in strong Kleene logic.

**Definition thr.2.** Halldén’s *logic of nonsense* \( Hal \) is defined using the matrix:

1. The standard propositional language \( L_0 \) with \( \neg, \wedge, \vee, \rightarrow \) and a 1-place connective \( + \).
2. The set of truth values \( V = \{ T, U, F \} \).
3. \( T \) and \( U \) are designated, i.e., \( V^+ = \{ T, U \} \).
4. Truth functions are the same as weak Kleene logic, plus the “is meaningless” operator:

\[
\begin{array}{c|ccc}
+ & T & U & F \\
\hline
T & T & T & F \\
U & T & T & F \\
F & F & F & F \\
\end{array}
\]

By contrast to the Kleene logics with which they share truth tables, these do have tautologies.

**Proposition thr.3.** The tautologies of \( LP \) are the same as the tautologies of classical propositional logic.

**Proof.** By ??, if \( \models_{LP} \varphi \) then \( \models_{C} \varphi \). To show the reverse, we show that if there is a valuation \( v : At_0 \rightarrow \{ F, T, U \} \) such that \( v_{Ks}(\varphi) = F \) then there is a valuation \( v' : At_0 \rightarrow \{ F, T \} \) such that \( v'_{C}(\varphi) = F \). This establishes the result for \( LP \), since \( Ks \) and \( LP \) have the same characteristic truth functions, and \( F \) is the only truth value of \( LP \) that is not designated (that is the only difference between \( LP \) and \( Ks \)). Thus, if \( \notmodels_{LP} \varphi \), for some valuation \( v \), \( v_{LP}(\varphi) = v_{Ks}(\varphi) = F \). By the claim we’re proving, \( v'_{C}(\varphi) = F \), i.e., \( \notmodels_{C} \varphi \).

To establish the claim, we first define \( v' \) as

\[
v'(p) = \begin{cases} 
T & \text{if } v(p) \in \{ T, U \} \\
F & \text{otherwise}
\end{cases}
\]
We now show by induction on \( \varphi \) that (a) if \( \mathfrak{v}_{Ks}(\varphi) = \mathcal{F} \) then \( \mathfrak{v}'_{C}(\varphi) = \mathcal{F} \), and (b) if \( \mathfrak{v}_{Ks}(\varphi) = \mathcal{T} \) then \( \mathfrak{v}'_{C}(\varphi) = \mathcal{T} \).

1. Induction basis: \( \varphi \equiv p \). By ??, \( \mathfrak{v}_{Ks}(\varphi) = \mathfrak{v}(p) = \mathfrak{v}'_{C}(\varphi) \), which implies both (a) and (b).

For the induction step, consider the cases:

2. \( \varphi \equiv \lnot \psi \).
   a) Suppose \( \mathfrak{v}_{Ks}(\lnot \psi) = \mathcal{F} \). By the definition of \( \lnot \mathfrak{Ks} \), \( \mathfrak{v}_{Ks}(\psi) = \mathcal{T} \). By inductive hypothesis, case (b), we get \( \mathfrak{v}'_{C}(\psi) = \mathcal{T} \), so \( \mathfrak{v}'_{C}(\lnot \psi) = \mathcal{F} \).
   b) Suppose \( \mathfrak{v}_{Ks}(\lnot \psi) = \mathcal{T} \). By the definition of \( \lnot \mathfrak{Ks} \), \( \mathfrak{v}_{Ks}(\psi) = \mathcal{F} \). By inductive hypothesis, case (a), we get \( \mathfrak{v}'_{C}(\psi) = \mathcal{F} \), so \( \mathfrak{v}'_{C}(\lnot \psi) = \mathcal{T} \).

3. \( \varphi \equiv (\psi \land \chi) \).
   a) Suppose \( \mathfrak{v}_{Ks}(\psi \land \chi) = \mathcal{F} \). By the definition of \( \land \mathfrak{Ks} \), \( \mathfrak{v}_{Ks}(\psi) = \mathcal{F} \) or \( \mathfrak{v}_{Ks}(\psi) = \mathcal{F} \). By inductive hypothesis, case (a), we get \( \mathfrak{v}'_{C}(\psi) = \mathcal{F} \) or \( \mathfrak{v}'_{C}(\chi) = \mathcal{F} \), so \( \mathfrak{v}'_{C}(\psi \land \chi) = \mathcal{F} \).
   b) Suppose \( \mathfrak{v}_{Ks}(\psi \land \chi) = \mathcal{T} \). By the definition of \( \land \mathfrak{Ks} \), \( \mathfrak{v}_{Ks}(\psi) = \mathcal{T} \) and \( \mathfrak{v}_{Ks}(\chi) = \mathcal{T} \). By inductive hypothesis, case (b), we get \( \mathfrak{v}'_{C}(\psi) = \mathcal{T} \) and \( \mathfrak{v}'_{C}(\chi) = \mathcal{T} \), so \( \mathfrak{v}'_{C}(\psi \land \chi) = \mathcal{T} \).

The other two cases are similar, and left as exercises. Alternatively, the proof above establishes the result for all formulas only containing \( \lnot \) and \( \land \). One may now appeal to the facts that in both \( Ks \) and \( C \), for any \( \mathfrak{v} \), \( \mathfrak{v}(\psi \lor \chi) = \mathfrak{v}(\lnot(\lnot \psi \land \lnot \chi)) \) and \( \mathfrak{v}(\psi \rightarrow \chi) = \mathfrak{v}(\lnot(\lnot \psi \land \lnot \chi)) \).

**Problem thr.1.** Complete the proof Proposition thr.3, i.e., establish (a) and (b) for the cases where \( \varphi \equiv (\psi \lor \chi) \) and \( \varphi \equiv (\psi \rightarrow \chi) \).

**Problem thr.2.** Prove that every classical tautology is a tautology in \( \text{Hal} \).

Although they have the same tautologies as classical logic, their consequence relations are different. \( LP \), for instance, is paraconsistent in that \( \lnot p, p \not \models q \), and so the principle of explosion \( \lnot \varphi, \varphi \models \psi \) does not hold in general. (It holds for some cases of \( \varphi \) and \( \psi \), e.g., if \( \psi \) is a tautology.)

**Problem thr.3.** Which of the following relations hold in (a) \( LP \) and in (b) \( \text{Hal} \)?

Give a truth table for each.

1. \( p, p \rightarrow q \models q \)
2. \( \lnot q, p \rightarrow q \models \lnot p \)
3. \( p \lor q, \lnot p \models q \)
4. \( \lnot p, p \models q \)
5. \( p \models p \lor q \)
6. \( p \rightarrow q, q \rightarrow r \models p \rightarrow r \)

What if you make \( U \) designated in \( L_3 \)?

**Definition thr.4.** The logic 3-valued R-Mingle \( \text{RM}_3 \) is defined using the matrix:

1. The standard propositional language \( \mathcal{L}_0 \) with \( \bot, \neg, \land, \lor, \rightarrow \).
2. The set of truth values \( V = \{T, U, F\} \).
3. \( T \) and \( U \) are designated, i.e., \( V^+ = \{T, U\} \).
4. Truth functions are the same as Łukasiewicz logic \( \text{L}_3 \).

**Problem thr.4.** Which of the following relations hold in \( \text{RM}_3 \)?

1. \( p, p \rightarrow q \models q \)
2. \( p \lor q, \neg p \models q \)
3. \( \neg p, p \models q \)
4. \( p \models p \lor q \)

Different truth tables can sometimes generate the same logic (entailment relation) just by changing the designated values. E.g., this happens if in Gödel logic we take \( V^+ = \{T, U\} \) instead of \( \{T\} \).

**Proposition thr.5.** The matrix with \( V = \{F, U, T\} \), \( V^+ = \{T, U\} \), and the truth functions of 3-valued Gödel logic defines classical logic.

**Proof.** Exercise. \( \square \)

**Problem thr.5.** Prove Proposition thr.5 by showing that for the logic \( \text{L} \) defined just like Gödel logic but with \( V^+ = \{T, U\} \), if \( \Gamma \not\vdash_\text{L} \psi \) then \( \Gamma \not\vdash_\text{C} \psi \). Use the ideas of Proposition thr.3, except instead of proving properties (a) and (b), show that \( \overline{v}_G(\varphi) = F \) iff \( \overline{v}_C(\varphi) = F \) (and hence that \( \overline{v}_G(\varphi) \in \{T, U\} \) iff \( \overline{v}_C(\varphi) = T \)). Explain why this establishes the proposition.

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**Bibliography**