Designating not just $T$

So far the logics we’ve seen all had the set of designated truth values $V^+ = \{T\}$, i.e., something counts as true iff its truth value is $T$. But one might also count something as true if it’s just not $F$. Then one would get a logic by stipulating in the matrix, e.g., that $V^+ = \{T, U\}$.

**Definition thr.1.** The *logic of paradox LP* is defined using the matrix:

1. The standard propositional language $L_0$ with $\neg, \land, \lor, \rightarrow$.
2. The set of truth values $V = \{T, U, F\}$.
3. $T$ and $U$ are designated, i.e., $V^+ = \{T, U\}$.
4. Truth functions are the same as in strong Kleene logic.

**Definition thr.2.** Halldén’s *logic of nonsense Hal* is defined using the matrix:

1. The standard propositional language $L_0$ with $\neg, \land, \lor, \rightarrow$ and a 1-place connective $\cdot$.
2. The set of truth values $V = \{T, U, F\}$.
3. $T$ and $U$ are designated, i.e., $V^+ = \{T, U\}$.
4. Truth functions are the same as weak Kleene logic, plus the “is meaningless” operator:

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By contrast to the Kleene logics with which they share truth tables, these *do* have tautologies.

**Proposition thr.3.** The tautologies of LP are the same as the tautologies of classical propositional logic.

*Proof.* By ??, if $\models_{\text{LP}} \varphi$ then $\models_{\text{C}} \varphi$. To show the reverse, we show that if there is a valuation $\nu$: $A_{l_0} \rightarrow \{F, T, U\}$ such that $\nu_{\text{KS}}(\varphi) = F$ then there is a valuation $\nu'$: $A_{l_0} \rightarrow \{F, T\}$ such that $\nu'_{\text{C}}(\varphi) = F$. This establishes the result for LP, since KSs and LP have the same characteristic truth functions, and $F$ is the only truth value of LP that is not designated (that is the only difference between LP and KSs). Thus, if $\not\models_{\text{LP}} \varphi$, for some valuation $\nu$, $\nu_{\text{LP}}(\varphi) = \nu_{\text{KS}}(\varphi) = F$. By the claim we’re proving, $\nu'_{\text{C}}(\varphi) = F$, i.e., $\not\models_{\text{C}} \varphi$.

To establish the claim, we first define $\nu'$ as

$$
\nu'(p) = \begin{cases} 
T & \text{if } \nu(p) \in \{T, U\} \\
F & \text{otherwise}
\end{cases}
$$
We now show by induction on $\varphi$ that (a) if $\bar{v}_K(\varphi) = F$ then $\bar{v}_C(\varphi) = F$, and (b) if $\bar{v}_K(\varphi) = T$ then $\bar{v}_C(\varphi) = T$

1. Induction basis: $\varphi \equiv p$. By ???, $\bar{v}_K(\varphi) = v(p) = \bar{v}_C(\varphi)$, which implies both (a) and (b).

For the induction step, consider the cases:

2. $\varphi \equiv \neg \psi$.

   a) Suppose $\bar{v}_K(\neg \psi) = F$. By the definition of $\neg K$, $\bar{v}_K(\psi) = T$. By inductive hypothesis, case (b), we get $\bar{v}_C(\psi) = T$, so $\bar{v}_C(\neg \psi) = F$.

   b) Suppose $\bar{v}_K(\neg \psi) = T$. By the definition of $\neg K$, $\bar{v}_K(\psi) = F$. By inductive hypothesis, case (a), we get $\bar{v}_C(\psi) = F$, so $\bar{v}_C(\neg \psi) = T$.

3. $\varphi \equiv (\psi \land \chi)$.

   a) Suppose $\bar{v}_K(\psi \land \chi) = F$. By the definition of $\land K$, $\bar{v}_K(\psi) = F$ or $\bar{v}_K(\chi) = F$. By inductive hypothesis, case (a), we get $\bar{v}_C(\psi) = F$ or $\bar{v}_C(\chi) = F$, so $\bar{v}_C(\psi \land \chi) = F$.

   b) Suppose $\bar{v}_K(\psi \land \chi) = T$. By the definition of $\land K$, $\bar{v}_K(\psi) = T$ and $\bar{v}_K(\chi) = T$. By inductive hypothesis, case (b), we get $\bar{v}_C(\psi) = T$ and $\bar{v}_C(\chi) = T$, so $\bar{v}_C(\psi \land \chi) = T$.

The other two cases are similar, and left as exercises. Alternatively, the proof above establishes the result for all formulas only containing $\neg$ and $\land$. One may now appeal to the facts that in both $K$ and $C$, for any $v$, $\bar{v}(\psi \lor \chi) = \bar{v}(\neg \neg \psi \land \neg \chi)$ and $\bar{v}(\psi \land \chi) = \bar{v}(\neg \neg \psi \land \neg \chi)$.

**Problem thr.1.** Complete the proof Proposition thr.3, i.e., establish (a) and (b) for the cases where $\varphi \equiv (\psi \lor \chi)$ and $\varphi \equiv (\psi \rightarrow \chi)$.

**Problem thr.2.** Prove that every classical tautology is a tautology in Hal.

Although they have the same tautologies as classical logic, their consequence relations are different. LP, for instance, is paraconsistent in that $\neg p, p \not\vdash q$, and so the principle of explosion $\neg \varphi, \varphi \vdash \psi$ does not hold in general. (It holds for some cases of $\varphi$ and $\psi$, e.g., if $\psi$ is a tautology.)

**Problem thr.3.** Which of the following relations hold in (a) LP and in (b) Hal? Give a truth table for each.

1. $p, p \rightarrow q \models q$
2. $\neg q, p \rightarrow q \not\models \neg p$
3. $p \lor q, \neg p \models q$
4. $\neg p, p \models q$
5. $p \models p \lor q$
6. \( p \rightarrow q, q \rightarrow r \models p \rightarrow r \)

What if you make \( U \) designated in \( L_3 \)?

**Definition thr.4.** The logic 3-valued R-Mingle \( RM_3 \) is defined using the matrix:

1. The standard propositional language \( L_0 \) with \( \bot, \neg, \land, \lor, \rightarrow \).
2. The set of truth values \( V = \{ T, U, F \} \).
3. \( T \) and \( U \) are designated, i.e., \( V^+ = \{ T, U \} \).
4. Truth functions are the same as Łukasiewicz logic \( L_3 \).

**Problem thr.4.** Which of the following relations hold in \( RM_3 \)?

1. \( p, p \rightarrow q \models q \)
2. \( p \lor q, \neg p \models q \)
3. \( \neg p, p \models q \)
4. \( p \models p \lor q \)

Different truth tables can sometimes generate the same logic (entailment relation) just by changing the designated values. E.g., this happens if in Gödel logic we take \( V^+ = \{ T, U \} \) instead of \( \{ T \} \).

**Proposition thr.5.** The matrix with \( V = \{ F, U, T \} \), \( V^+ = \{ T, U \} \), and the truth functions of 3-valued Gödel logic defines classical logic.

**Proof.** Exercise.

**Problem thr.5.** Prove Proposition thr.5 by showing that for the logic \( L \) defined just like Gödel logic but with \( V^+ = \{ T, U \} \), if \( \Gamma \not\models_L \psi \) then \( \Gamma \not\models_C \psi \). Use the ideas of Proposition thr.3, except instead of proving properties (a) and (b), show that \( v_G(\phi) = F \) iff \( v_C(\phi) = F \) (and hence that \( v_G(\phi) \in \{ T, U \} \) iff \( v_C(\phi) = T \)). Explain why this establishes the proposition.

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**Bibliography**