

Chapter udf

Syntax and Semantics

syn.1 Introduction

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In classical logic, we deal with **formulas** that are built from **propositional variables** using the propositional connectives \neg , \wedge , \vee , \rightarrow , and \leftrightarrow . When we define a semantics for classical logic, we do so using the two truth values \mathbb{T} and \mathbb{F} . We interpret **propositional variables** in a **valuation** \mathbf{v} , which assigns these truth values \mathbb{T} , \mathbb{F} to the **propositional variables**. Any **valuation** then determines a truth value $\bar{\mathbf{v}}(\varphi)$ for any **formula** φ , and **A formula** is satisfied in a **valuation** \mathbf{v} , $\mathbf{v} \models \varphi$, iff $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$.

Many-valued logics are generalizations of classical two-valued logic by allowing more truth values than just \mathbb{T} and \mathbb{F} . So in many-valued logic, a **valuation** \mathbf{v} is a function assigning to every **propositional variable** p one of a range of possible truth values. We'll generally call the set of allowed truth values V . Classical logic is a many-valued logic where $V = \{\mathbb{T}, \mathbb{F}\}$, and the truth value $\bar{\mathbf{v}}(\varphi)$ is computed using the familiar characteristic truth tables for the connectives.

Once we add additional truth values, we have more than one natural option for how to compute $\bar{\mathbf{v}}(\varphi)$ for the connectives we read as “and,” “or,” “not,” and “if—then.” So a many-valued logic is determined not just by the set of truth values, but also by the *truth functions* we decide to use for each connective. Once these are selected for a many-valued logic \mathbf{L} , however, the truth value $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi)$ is uniquely determined by the valuation, just like in classical logic. Many-valued logics, like classical logic, are *truth functional*.

With this semantic building blocks in hand, we can go on to define the analogs of the semantic concepts of tautology, entailment, and satisfiability. In classical logic, a **formula** is a tautology if its truth value $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ for any \mathbf{v} . In many-valued logic, we have to generalize this a bit as well. First of all, there is no requirement that the set of truth values V contains \mathbb{T} . For instance, some many-valued logics use numbers, such as all rational numbers between 0 and 1 as their set of truth values. In such a case, 1 usually plays the role of \mathbb{T} . In other logics, not just one but several truth values do. So, we require that every many-valued logic have a set V^+ of *designated values*. We can then say that

a **formula** is satisfied in a **valuation** \mathbf{v} , $\mathbf{v} \models_{\mathbf{L}} \varphi$, iff $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) \in V^+$. A **formula** φ is a tautology of the logic, $\models_{\mathbf{L}} \varphi$, iff $\bar{\mathbf{v}}(\varphi) \in V^+$ for any \mathbf{v} . And, finally, we say that φ is entailed by a set of **formulas**, $\Gamma \models_{\mathbf{L}} \varphi$, if every **valuation** that satisfies all the **formulas** in Γ also satisfies φ .

syn.2 Languages and Connectives

Classical propositional logic, and many other logics, use a set supply of *propositional constants* and *connectives*. For instance, we use the following as primitives:

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1. The propositional constant for **falsity** \perp .
2. The propositional constant for **truth** \top .
3. The logical connectives: \neg (negation), \wedge (conjunction), \vee (disjunction), \rightarrow (**conditional**), \leftrightarrow (**biconditional**)

The same connectives are used in many-valued logics as well. However, it is often useful to include different versions of, say, conjunction, in the same logic, and that would require different symbols to keep the versions separate. Some many-valued logics also include connectives that have no equivalent in classical logic. So, we'll be a bit more general than usual.

Definition syn.1. A *propositional language* consists of a set \mathcal{L} of *connectives*. Each connective \star has an *arity*; a connective of arity n is said to be *n-place*. Connectives of arity 0 are also called *constants*; connectives of arity 1 are called *unary*, and connectives of arity 2, *binary*.

Example syn.2. The standard language of propositional logic \mathcal{L}_0 consists of the following connectives (with associated arities): \perp (0) \neg (1), \wedge (2), \vee (2), \rightarrow (2). Most logics we consider will use this language. Some logics by tradition an convention use different symbols for some connectives. For instance, in product logic, the conjunction symbol is often \odot instead of \wedge . Sometimes it is convenient to add a new operator, e.g., the determinateness operator Δ (1-place).

syn.3 Formulas

Definition syn.3 (Formula). The set $\text{Frm}(\mathcal{L})$ of *formulas* of a propositional language \mathcal{L} is defined inductively as follows:

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mvl:syn:fml:
defn:formulas

1. Every **propositional variable** p_i is an atomic **formula**.
2. Every 0-place connective (propositional constant) of \mathcal{L} is an atomic **formula**.

3. If \star is an n -place connective of \mathcal{L} , and $\varphi_1, \dots, \varphi_n$ are formulas, then $\star(\varphi_1, \dots, \varphi_n)$ is a formula.
4. Nothing else is a formula.

If \star is 1-place, then $\star(\varphi_1)$ will often be written simply as $\star\varphi_1$. If \star is 2-place $\star(\varphi_1, \varphi_2)$ will often be written as $(\varphi_1 \star \varphi_2)$.

As usual, we will often silently leave out the outermost parentheses.

Example syn.4. In the standard language \mathcal{L}_0 , $p_1 \rightarrow (p_1 \wedge \neg p_2)$ is a formula. In the language of product logic, it would be written instead as $p_1 \rightarrow (p_1 \odot \neg p_2)$. If we add the 1-place Δ to the language, we would also have formulas such as $\Delta(p_1 \wedge p_2) \rightarrow (\Delta p_1 \wedge \Delta p_2)$.

syn.4 Matrices

mvl:syn:mat:
sec A many-valued logic is defined by its language, its set of truth values V , a subset of designated truth values, and truth functions for its connective. Together, these elements are called a *matrix*.

mvl:syn:mat:
defn:matrix **Definition syn.5 (Matrix).** A *matrix* for the logic \mathbf{L} consists of:

1. a set of connectives making up a language \mathcal{L} ;
2. a set $V \neq \emptyset$ of truth values;
3. a set $V^+ \subseteq V$ of designated truth values;
4. for each n -place connective \star in \mathcal{L} , a truth function $\tilde{\star} : V^n \rightarrow V$. If $n = 0$, then $\tilde{\star}$ is just an element of V .

Example syn.6. The matrix for classical logic \mathbf{C} consists of:

1. The standard propositional language \mathcal{L}_0 with $\perp, \neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. For \perp , we have $\tilde{\perp} = \mathbb{F}$. The other truth functions are given by the usual truth tables (see [Figure syn.1](#)).

\simeq		$\tilde{\wedge}$	T	F	$\tilde{\vee}$	T	F	$\tilde{\supset}$	T	F
T	F	T	T	F	T	T	T	T	T	F
F	T	F	F	F	F	T	F	F	T	T

Figure syn.1: Truth functions for classical logic **C**.

mvl:syn:mat:
fig:tf-CL

syn.5 Valuations and Satisfaction

Definition syn.7 (Valuations). Let V be a set of truth values. A *valuation* for \mathcal{L} into V is a function \mathbf{v} assigning an element of V to the propositional variables of the language, i.e., $\mathbf{v}: \text{At}_0 \rightarrow V$.

mvl:syn:val:
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Definition syn.8. Given a valuation \mathbf{v} into the set of truth values V of a many-valued logic \mathbf{L} , define the evaluation function $\bar{\mathbf{v}}: \text{Frm}(\mathcal{L}) \rightarrow V$ inductively by:

mvl:syn:val:
defn:pValue

1. $\bar{\mathbf{v}}(\rho_n) = \mathbf{v}(\rho_n)$;
2. If \star is a 0-place connective, then $\bar{\mathbf{v}}(\star) = \tilde{\star}_{\mathbf{L}}$;
3. If \star is an n -place connective, then

$$\bar{\mathbf{v}}(\star(\varphi_1, \dots, \varphi_n)) = \tilde{\star}_{\mathbf{L}}(\bar{\mathbf{v}}(\varphi_1), \dots, \bar{\mathbf{v}}(\varphi_n)).$$

Definition syn.9 (Satisfaction). The formula φ is *satisfied* by a valuation \mathbf{v} , $\mathbf{v} \models_{\mathbf{L}} \varphi$, iff $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) \in V^+$, where V^+ is the set of designated truth values of \mathbf{L} .

mvl:syn:val:
defn:satisfaction

We write $\mathbf{v} \not\models_{\mathbf{L}} \varphi$ to mean “not $\mathbf{v} \models_{\mathbf{L}} \varphi$.” If Γ is a set of formulas, $\mathbf{v} \models_{\mathbf{L}} \Gamma$ iff $\mathbf{v} \models_{\mathbf{L}} \varphi$ for every $\varphi \in \Gamma$.

syn.6 Semantic Notions

Suppose a many-valued logic \mathbf{L} is given by a matrix. Then we can define the usual semantic notions for \mathbf{L} .

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Definition syn.10. 1. A formula φ is *satisfiable* if for some \mathbf{v} , $\mathbf{v} \models \varphi$; it is *unsatisfiable* if for no \mathbf{v} , $\mathbf{v} \models \varphi$;

2. A formula φ is a *tautology* if $\mathbf{v} \models \varphi$ for all valuations \mathbf{v} ;
3. If Γ is a set of formulas, $\Gamma \models \varphi$ (“ Γ entails φ ”) if and only if $\mathbf{v} \models \varphi$ for every valuation \mathbf{v} for which $\mathbf{v} \models \Gamma$.
4. If Γ is a set of formulas, Γ is *satisfiable* if there is a valuation \mathbf{v} for which $\mathbf{v} \models \Gamma$, and Γ is *unsatisfiable* otherwise.

We have some of the same facts for these notions as we do for the case of classical logic:

mul:syn:sem:
prop:semanticalfacts

Proposition syn.11.

1. φ is a tautology if and only if $\emptyset \models \varphi$;
2. If Γ is satisfiable then every finite subset of Γ is also satisfiable;
3. Monotonicity: if $\Gamma \subseteq \Delta$ and $\Gamma \models \varphi$ then also $\Delta \models \varphi$;
4. Transitivity: if $\Gamma \models \varphi$ and $\Delta \cup \{\varphi\} \models \psi$ then $\Gamma \cup \Delta \models \psi$;

mul:syn:sem:
def:monotonicity

mul:syn:sem:
def:Cut

Proof. Exercise. □

Problem syn.1. Prove **Proposition syn.11**

In classical logic we can connect entailment and the conditional. For instance, we have the validity of *modus ponens*: If $\Gamma \models \varphi$ and $\Gamma \models \varphi \rightarrow \psi$ then $\Gamma \models \psi$. Another important relationship between \models and \rightarrow in classical logic is the semantic deduction theorem: $\Gamma \models \varphi \rightarrow \psi$ if and only if $\Gamma \cup \{\varphi\} \models \psi$. These results *do not* always hold in many-valued logics. Whether they do depends on the truth function $\widetilde{\rightarrow}$.

syn.7 Many-valued logics as sublogics of C

mul:syn:sub:
sec

The usual many-valued logics are all defined using matrices in which the value of a truth-function for arguments in $\{\mathbb{T}, \mathbb{F}\}$ agrees with the classical truth functions. Specifically, in these logics, if $x \in \{\mathbb{T}, \mathbb{F}\}$, then $\widetilde{\neg}_{\mathbf{L}}(x) = \widetilde{\neg}_{\mathbf{C}}(x)$, and for \star any one of $\wedge, \vee, \rightarrow$, if $x, y \in \{\mathbb{T}, \mathbb{F}\}$, then $\widetilde{\star}_{\mathbf{L}}(x, y) = \widetilde{\star}_{\mathbf{C}}(x, y)$. In other words, the truth functions for $\neg, \wedge, \vee, \rightarrow$ restricted to $\{\mathbb{T}, \mathbb{F}\}$ are exactly the classical truth functions.

mul:syn:sub:
prop:mul-cl

Proposition syn.12. *Suppose that a many-valued logic \mathbf{L} contains the connectives $\neg, \wedge, \vee, \rightarrow$ in its language, $\mathbb{T}, \mathbb{F} \in V$, and its truth functions satisfy:*

mul:syn:sub:
prop:not

1. $\widetilde{\neg}_{\mathbf{L}}(x) = \widetilde{\neg}_{\mathbf{C}}(x)$ if $x = \mathbb{T}$ or $x = \mathbb{F}$;

mul:syn:sub:
prop:land

2. $\widetilde{\wedge}_{\mathbf{L}}(x, y) = \widetilde{\wedge}_{\mathbf{C}}(x, y)$,

mul:syn:sub:
prop:lor

3. $\widetilde{\vee}_{\mathbf{L}}(x, y) = \widetilde{\vee}_{\mathbf{C}}(x, y)$,

mul:syn:sub:
prop:lif

4. $\widetilde{\rightarrow}_{\mathbf{L}}(x, y) = \widetilde{\rightarrow}_{\mathbf{C}}(x, y)$, if $x, y \in \{\mathbb{T}, \mathbb{F}\}$.

Then, for any valuation \mathbf{v} into V such that $\mathbf{v}(p) \in \{\mathbb{T}, \mathbb{F}\}$, $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \bar{\mathbf{v}}_{\mathbf{C}}(\varphi)$.

Proof. By induction on φ .

1. If $\varphi \equiv p$ is atomic, we have $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \mathbf{v}(p) = \bar{\mathbf{v}}_{\mathbf{C}}(\varphi)$.

2. If $\varphi \equiv \neg B$, we have

$$\begin{aligned}
\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) &= \tilde{\neg}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{L}}(\psi)) && \text{by Definition syn.8} \\
&= \tilde{\neg}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi)) && \text{by inductive hypothesis} \\
&= \tilde{\neg}_{\mathbf{C}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi)) && \text{by assumption (1),} \\
& && \text{since } \bar{\mathbf{v}}_{\mathbf{C}}(\psi) \in \{\mathbb{T}, \mathbb{F}\}, \\
&= \bar{\mathbf{v}}_{\mathbf{C}}(\varphi) && \text{by Definition syn.8.}
\end{aligned}$$

3. If $\varphi \equiv (\psi \wedge \chi)$, we have

$$\begin{aligned}
\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) &= \tilde{\wedge}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{L}}(\psi), \bar{\mathbf{v}}_{\mathbf{L}}(\chi)) && \text{by Definition syn.8} \\
&= \tilde{\wedge}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi)) && \text{by inductive hypothesis} \\
&= \tilde{\wedge}_{\mathbf{C}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi)) && \text{by assumption (2),} \\
& && \text{since } \bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi) \in \{\mathbb{T}, \mathbb{F}\}, \\
&= \bar{\mathbf{v}}_{\mathbf{C}}(\varphi) && \text{by Definition syn.8.}
\end{aligned}$$

The cases where $\varphi \equiv (\psi \vee \chi)$ and $\varphi \equiv (\psi \rightarrow \chi)$ are similar. \square

Corollary syn.13. *If a many-valued logic satisfies the conditions of Proposition syn.12, $\mathbb{T} \in V^+$ and $\mathbb{F} \notin V^+$, then $\models_{\mathbf{L}} \subseteq \models_{\mathbf{C}}$, i.e., if $\Gamma \models_{\mathbf{L}} \psi$ then $\Gamma \models_{\mathbf{C}} \psi$. In particular, every tautology of \mathbf{L} is also a classical tautology.*

Proof. We prove the contrapositive. Suppose $\Gamma \not\models_{\mathbf{C}} \psi$. Then there is some valuation $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{T}, \mathbb{F}\}$ such that $\bar{\mathbf{v}}_{\mathbf{C}}(\varphi) = \mathbb{T}$ for all $\varphi \in \Gamma$ and $\bar{\mathbf{v}}_{\mathbf{C}}(\psi) = \mathbb{F}$. Since $\mathbb{T}, \mathbb{F} \in V$, the valuation \mathbf{v} is also a valuation for \mathbf{L} . By Proposition syn.12, $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \mathbb{T}$ for all $\varphi \in \Gamma$ and $\bar{\mathbf{v}}_{\mathbf{L}}(\psi) = \mathbb{F}$. Since $\mathbb{T} \in V^+$ and $\mathbb{F} \notin V^+$ that means $\mathbf{v} \models_{\mathbf{L}} \Gamma$ and $\mathbf{v} \not\models_{\mathbf{L}} \psi$, i.e., $\Gamma \not\models_{\mathbf{L}} \psi$. \square

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Bibliography