

syn.1 Many-valued logics as sublogics of \mathbf{C}

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sec

The usual many-valued logics are all defined using matrices in which the value of a truth-function for arguments in $\{\mathbb{T}, \mathbb{F}\}$ agrees with the classical truth functions. Specifically, in these logics, if $x \in \{\mathbb{T}, \mathbb{F}\}$, then $\tilde{\neg}_{\mathbf{L}}(x) = \tilde{\neg}_{\mathbf{C}}(x)$, and for \star any one of $\wedge, \vee, \rightarrow$, if $x, y \in \{\mathbb{T}, \mathbb{F}\}$, then $\tilde{\star}_{\mathbf{L}}(x, y) = \tilde{\star}_{\mathbf{C}}(x, y)$. In other words, the truth functions for $\neg, \wedge, \vee, \rightarrow$ restricted to $\{\mathbb{T}, \mathbb{F}\}$ are exactly the classical truth functions.

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Proposition syn.1. *Suppose that a many-valued logic \mathbf{L} contains the connectives $\neg, \wedge, \vee, \rightarrow$ in its language, $\mathbb{T}, \mathbb{F} \in V$, and its truth functions satisfy:*

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1. $\tilde{\neg}_{\mathbf{L}}(x) = \tilde{\neg}_{\mathbf{C}}(x)$ if $x = \mathbb{T}$ or $x = \mathbb{F}$;

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2. $\tilde{\wedge}_{\mathbf{L}}(x, y) = \tilde{\wedge}_{\mathbf{C}}(x, y)$,

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3. $\tilde{\vee}_{\mathbf{L}}(x, y) = \tilde{\vee}_{\mathbf{C}}(x, y)$,

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4. $\tilde{\rightarrow}_{\mathbf{L}}(x, y) = \tilde{\rightarrow}_{\mathbf{C}}(x, y)$, if $x, y \in \{\mathbb{T}, \mathbb{F}\}$.

Then, for any valuation \mathbf{v} into V such that $\mathbf{v}(p) \in \{\mathbb{T}, \mathbb{F}\}$, $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \bar{\mathbf{v}}_{\mathbf{C}}(\varphi)$.

Proof. By induction on φ .

1. If $\varphi \equiv p$ is atomic, we have $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \mathbf{v}(p) = \bar{\mathbf{v}}_{\mathbf{C}}(\varphi)$.

2. If $\varphi \equiv \neg B$, we have

$$\begin{aligned} \bar{\mathbf{v}}_{\mathbf{L}}(\varphi) &= \tilde{\neg}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{L}}(\psi)) && \text{by ??} \\ &= \tilde{\neg}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi)) && \text{by inductive hypothesis} \\ &= \tilde{\neg}_{\mathbf{C}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi)) && \text{by assumption (1),} \\ & && \text{since } \bar{\mathbf{v}}_{\mathbf{C}}(\psi) \in \{\mathbb{T}, \mathbb{F}\}, \\ &= \bar{\mathbf{v}}_{\mathbf{C}}(\varphi) && \text{by ??} \end{aligned}$$

3. If $\varphi \equiv (\psi \wedge \chi)$, we have

$$\begin{aligned} \bar{\mathbf{v}}_{\mathbf{L}}(\varphi) &= \tilde{\wedge}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{L}}(\psi), \bar{\mathbf{v}}_{\mathbf{L}}(\chi)) && \text{by ??} \\ &= \tilde{\wedge}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi)) && \text{by inductive hypothesis} \\ &= \tilde{\wedge}_{\mathbf{C}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi)) && \text{by assumption (2),} \\ & && \text{since } \bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi) \in \{\mathbb{T}, \mathbb{F}\}, \\ &= \bar{\mathbf{v}}_{\mathbf{C}}(\varphi) && \text{by ??} \end{aligned}$$

The cases where $\varphi \equiv (\psi \vee \chi)$ and $\varphi \equiv (\psi \rightarrow \chi)$ are similar. \square

Corollary syn.2. *If a many-valued logic satisfies the conditions of [Proposition syn.1](#), $\mathbb{T} \in V^+$ and $\mathbb{F} \notin V^+$, then $\models_{\mathbf{L}} \subseteq \models_{\mathbf{C}}$, i.e., if $\Gamma \models_{\mathbf{L}} \psi$ then $\Gamma \models_{\mathbf{C}} \psi$. In particular, every tautology of \mathbf{L} is also a classical tautology.*

Proof. We prove the contrapositive. Suppose $\Gamma \not\vdash_{\mathbf{C}} \psi$. Then there is some valuation $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{T}, \mathbb{F}\}$ such that $\bar{\mathbf{v}}_{\mathbf{C}}(\varphi) = \mathbb{T}$ for all $\varphi \in \Gamma$ and $\bar{\mathbf{v}}_{\mathbf{C}}(\psi) = \mathbb{F}$. Since $\mathbb{T}, \mathbb{F} \in V$, the valuation \mathbf{v} is also a valuation for \mathbf{L} . By Proposition syn.1, $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \mathbb{T}$ for all $\varphi \in \Gamma$ and $\bar{\mathbf{v}}_{\mathbf{L}}(\psi) = \mathbb{F}$. Since $\mathbb{T} \in V^+$ and $\mathbb{F} \notin V^+$ that means $\mathbf{v} \models_{\mathbf{L}} \Gamma$ and $\mathbf{v} \not\models_{\mathbf{L}} \psi$, i.e., $\Gamma \not\vdash_{\mathbf{L}} \psi$. \square

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Bibliography