Many-valued logics as sublogics of $C$

The usual many-valued logics are all defined using matrices in which the value of a truth-function for arguments in $\{T, F\}$ agrees with the classical truth functions. Specifically, in these logics, if $x \in \{T, F\}$, then $\neg_L(x) = \neg_C(x)$, and for any one of $\land, \lor, \rightarrow$, if $x, y \in \{T, F\}$, then $\ast_L(x, y) = \ast_C(x, y)$. In other words, the truth functions for $\neg, \land, \lor, \rightarrow$ restricted to $\{T, F\}$ are exactly the classical truth functions.

**Proposition syn.1.** Suppose that a many-valued logic $L$ contains the connectives $\neg, \land, \lor, \rightarrow$ in its language, $T, F \in V$, and its truth functions satisfy:

1. $\neg_L(x) = \neg_C(x)$ if $x = T$ or $x = F$;
2. $\land_L(x, y) = \land_C(x, y)$,
3. $\lor_L(x, y) = \lor_C(x, y)$,
4. $\rightarrow_L(x, y) = \rightarrow_C(x, y)$, if $x, y \in \{T, F\}$.

Then, for any valuation $v$ into $V$ such that $v(p) \in \{T, F\}$, $v_L(\varphi) = v_C(\varphi)$.

**Proof.** By induction on $\varphi$.

1. If $\varphi \equiv p$ is atomic, we have $v_L(\varphi) = v(p) = v_C(\varphi)$.
2. If $\varphi \equiv \neg B$, we have

   $v_L(\varphi) = \neg_L(v_L(\psi))$
   $= \neg_L(v_C(\psi))$
   $= \neg_C(v_C(\psi))$
   $= v_C(\varphi)$

   by ??

   by inductive hypothesis

   by assumption (1),

   since $v_C(\psi) \in \{T, F\}$,

   by ??.

3. If $\varphi \equiv (\psi \land \chi)$, we have

   $v_L(\varphi) = \land_L(v_L(\psi), v_L(\chi))$
   $= \land_L(v_C(\psi), v_C(\chi))$
   $= \land_C(v_C(\psi), v_C(\chi))$
   $= v_C(\varphi)$

   by ??

   by inductive hypothesis

   by assumption (2),

   since $v_C(\psi), v_C(\chi) \in \{T, F\}$,

   by ??.

The cases where $\varphi \equiv (\psi \lor \chi)$ and $\varphi \equiv (\psi \rightarrow \chi)$ are similar.

**Corollary syn.2.** If a many-valued logic satisfies the conditions of Proposition syn.1, $T \in V^+$ and $F \not\in V^+$, then $\models_L \psi$ if $\models_C \psi$. In particular, every tautology of $L$ is also a classical tautology.
Proof. We prove the contrapositive. Suppose $\Gamma \nvDash_C \psi$. Then there is some valuation $v: A_{t_0} \to \{T, F\}$ such that $v_C(\varphi) = T$ for all $\varphi \in \Gamma$ and $v_C(\psi) = F$. Since $T, F \in V$, the valuation $v$ is also a valuation for $L$. By Proposition syn.1, $v_L(\varphi) = T$ for all $\varphi \in \Gamma$ and $v_L(\psi) = F$. Since $T \in V^+$ and $F \notin V^+$ that means $v \models L \Gamma$ and $v \not\models L \psi$, i.e., $\Gamma \nvDash_L \psi$. \qed

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Bibliography