Many-valued logics as sublogics of $C$

The usual many-valued logics are all defined using matrices in which the value of a truth-function for arguments in $\{T, F\}$ agrees with the classical truth functions. Specifically, in these logics, if $x \in \{T, F\}$, then $\neg_L(x) = \neg_C(x)$, and for any one of $\land, \lor, \to$, if $x, y \in \{T, F\}$, then $\star_L(x, y) = \star_C(x, y)$. In other words, the truth functions for $\neg, \land, \lor, \to$ restricted to $\{T, F\}$ are exactly the classical truth functions.

Proposition syn.1. Suppose that a many-valued logic $L$ contains the connectives $\neg, \land, \lor, \to$ in its language, $T, F \in V$, and its truth functions satisfy:

1. $\neg_L(x) = \neg_C(x)$ if $x = T$ or $x = F$;
2. $\land_L(x, y) = \land_C(x, y)$,
3. $\lor_L(x, y) = \lor_C(x, y)$,
4. $\to_L(x, y) = \to_C(x, y)$, if $x, y \in \{T, F\}$.

Then, for any valuation $v$ into $V$ such that $v(p) \in \{T, F\}$, $v_L(\varphi) = v_C(\varphi)$.

Proof. By induction on $\varphi$.

1. If $\varphi \equiv p$ is atomic, we have $v_L(\varphi) = v(p) = v_C(\varphi)$.

2. If $\varphi \equiv \neg B$, we have
   \begin{align*}
   v_L(\varphi) &= \neg_L(v_L(\psi)) \\
   &= \neg_L(v_C(\psi)) \quad \text{by } ?? \quad \text{by inductive hypothesis} \\
   &= \neg_C(v_C(\psi)) \quad \text{by assumption (1),} \\
   &= v_C(\varphi) \quad \text{since } v_C(\psi) \in \{T, F\}, \text{ by } ??.
   \end{align*}

3. If $\varphi \equiv (\psi \land \chi)$, we have
   \begin{align*}
   v_L(\varphi) &= \land_L(v_L(\psi), v_L(\chi)) \\
   &= \land_L(v_C(\psi), v_C(\chi)) \quad \text{by } ?? \quad \text{by inductive hypothesis} \\
   &= \land_C(v_C(\psi), v_C(\chi)) \quad \text{by assumption (2),} \\
   &= v_C(\varphi) \quad \text{since } v_C(\psi), v_C(\chi) \in \{T, F\}, \text{ by } ??.
   \end{align*}

The cases where $\varphi \equiv (\psi \lor \chi)$ and $\varphi \equiv (\psi \to \chi)$ are similar. \hfill \square

Corollary syn.2. If a many-valued logic satisfies the conditions of Proposition syn.1, $T \in V^+$ and $F \notin V^+$, then $\Gamma_L \models \psi$ if and only if $\Gamma_C \models \psi$, i.e., if $\Gamma_L \models \psi$ then $\Gamma_C \models \psi$. In particular, every tautology of $L$ is also a classical tautology.
Proof. We prove the contrapositive. Suppose $\Gamma \not\models_c \psi$. Then there is some valuation $v : \mathcal{A}_0 \rightarrow \{T, F\}$ such that $v_C(\varphi) = T$ for all $\varphi \in \Gamma$ and $v_C(\psi) = F$. Since $T, F \in V$, the valuation $v$ is also a valuation for $L$. By Proposition syn.1, $v_L(\varphi) = T$ for all $\varphi \in \Gamma$ and $v_L(\psi) = F$. Since $T \in V^+$ and $F \not\in V^+$ that means $v \models_L \Gamma$ and $v \not\models_L \psi$, i.e., $\Gamma \not\models_L \psi$. \qed

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Bibliography