# Chapter udf

# Sequent Calculus

## seq.1 Introduction

mvl:seq:int:sec The sequent calculus for classical logic is an efficient and simple derivation system. If a many-valued logic is defined by a matrix with finitely many truth values, i.e., $V$ is finite, it is possible to provide a sequent calculus for it. The idea for how to do this comes from considering the meanings of sequents and the form of inference rules in the classical case.

Now recall that a sequent

$$\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_n$$

can be interpreted as the formula

$$(\varphi_1 \land \cdots \land \varphi_m) \to (\psi_1 \lor \cdots \lor \psi_n)$$

In other words, A valuation $\mathfrak{v}$ *satisfies* a sequent $\Gamma \Rightarrow \Delta$ iff either $\overline{\mathfrak{v}}(\varphi) = \mathbb{F}$ for some $\varphi \in \Gamma$ or $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ for some $\varphi \in \Delta$. On this interpretation, initial sequents $\varphi \Rightarrow \varphi$ are always satisfied, because either $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ or $\overline{}(\varphi) = \mathbb{F}$.

Here are the inference rules for the conditional in **LK**, with side formulas $\Gamma$, $\Delta$ left out:

$$\dfrac{\Rightarrow \varphi \qquad \psi \Rightarrow}{\varphi \to \psi \Rightarrow} \to\text{L} \qquad\qquad \dfrac{\varphi \Rightarrow \psi}{\Rightarrow \varphi \to \psi} \to\text{R}$$

If we apply the above semantic interpretation of a sequent, we can read the $\to$L rule as saying that if $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ and $\overline{\mathfrak{v}}(\psi) = \mathbb{F}$, then $\overline{\mathfrak{v}}(\varphi \to \psi) = \mathbb{F}$. Similarly, the $\to$R rule says that if either $\overline{\mathfrak{v}}(\varphi) = \mathbb{F}$ or $\overline{\mathfrak{v}}(\psi) = \mathbb{T}$, then $\overline{\mathfrak{v}}(\varphi \to \psi) = \mathbb{T}$. And in fact, these conditionals are actually biconditionals. In the case of the $\land$L and $\lor$R rules in their standard formulation, the corresponding conditionals would not be biconditionals. But there are alternative versions of these rules where they are:

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee\mathrm{R}$$

This basic idea, applied to an $n$-valued logic, then results in a sequent calculus with $n$ instead of two places, one for each truth value. For a three-valued logic with $V = \{\mathbb{F}, \mathbb{U}, \mathbb{T}\}$, a sequent is an expression $\Gamma \mid \Pi \mid \Delta$. It is satisfied in a valuation $\mathfrak{v}$ iff either $\overline{\mathfrak{v}}(\varphi) = \mathbb{F}$ for some $\varphi \in \Gamma$ or $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ for some $\varphi \in \Delta$ or $\overline{\mathfrak{v}}(\varphi) = \mathbb{U}$ for some $\varphi \in \Pi$. Consequently, initial sequents $\varphi \mid \varphi \mid \varphi$ are always satisfied.

## seq.2 Rules and Derivations

For the following, let $\Gamma, \Delta, \Pi, \Lambda$ represent finite sequences of sentences. mvl:seq:rul:sec

**Definition seq.1 (Sequent).** An *$n$-sided sequent* is an expression of the form

$$\Gamma_1 \mid \ldots \mid \Gamma_n$$

where each $\Gamma_1$ is a finite (possibly empty) sequences of sentences of the language $\mathcal{L}$.

**Definition seq.2 (Initial Sequent).** An *$n$-sided initial sequent* is an $n$-sided sequent of the form $\varphi \mid \ldots \mid \varphi$ for any sentence $\varphi$ in the language.

If the language contains a 0-place connective $\star$, i.e., a propositional constant, then we also take the sequent $\ldots \mid \star \mid \ldots$ where $\star$ appears in the space for the truth value associated with $\widetilde{\star} \in V$, and is empty otherwise.

For each connective of an $n$-valued logic $\mathbf{L}$, there is a logical rule for each truth value that this connective can take in $\mathbf{L}$. Derivations in an $n$-sided sequent calculus for $\mathbf{L}$ are trees of sequents, where the topmost sequents are initial sequents, and if a sequent stands below one or more other sequents, it must follow correctly by a rule of inference for the connectives of $\mathbf{L}$.

**Definition seq.3 (Theorems).** A sentence $\varphi$ is a *theorem* of an $n$-valued logic $\mathbf{L}$ if there is a derivation of the $n$-sequent containing $\varphi$ in each position corresponding to a designated truth value of $\mathbf{L}$. We write $\vdash_{\mathbf{L}} \varphi$ if $\varphi$ is a theorem and $\nvdash_{\mathbf{L}} \varphi$ if it is not.

**Definition seq.4 (Derivability).** A sentence $\varphi$ is *derivable from* a set of sentences $\Gamma$ in an $n$-valued logic $\mathbf{L}$, $\Gamma \vdash_{\mathbf{L}} \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence $\Gamma_0'$ of the sentences in $\Gamma_0$ such that the following sequent has a derivation:

$$\Lambda_1 \mid \ldots \mid \Lambda_n$$

where $\Lambda_i$ is $\varphi$ if position $i$ corresponds to a designated truth value, and $\Gamma_0'$otherwise. If $\varphi$ is not derivable from $\Gamma$ we write $\Gamma \nvdash \varphi$.

For instance, 3-valued Łukasiewicz logic has a 3-sided sequent calculus. In a 3-sided sequent $\Gamma \mid \Pi \mid \Delta$, $\Gamma$ corresponds to $\mathbb{F}$, $\Delta$ to $\mathbb{T}$, and $\Pi$ to $\mathbb{U}$. Axioms are $\varphi \mid \varphi \mid \varphi$. Since only $\mathbb{T}$ is designated, $\Gamma \vdash_{\mathbf{Ł}_3} \varphi$ iff the sequent $\Gamma \mid \Gamma \mid \varphi$ has a derivation. (If $\mathbb{U}$ were also designated, we would need a derivation of $\Gamma \mid \varphi \mid \varphi$.)

## seq.3 Structural Rules

mvl:seq:str:sec The structural rules for $n$-sided sequent calculus operate as in the classical case, except for each position $i$.

$$\frac{\Gamma_1 \mid \dots \mid \quad \Gamma_i \mid \dots \mid \Gamma_n}{\Gamma_1 \mid \dots \mid \varphi, \Gamma_i \mid \dots \mid \Gamma_n}\,\mathrm{W}i$$

$$\frac{\Gamma_1 \mid \dots \mid \varphi, \varphi, \Gamma_i \mid \dots \mid \Gamma_n}{\Gamma_1 \mid \dots \mid \quad \varphi, \Gamma_i \mid \dots \mid \Gamma_n}\,\mathrm{C}i$$

$$\frac{\Gamma_1 \mid \dots \mid \Gamma_i, \varphi, \psi, \Gamma_i' \mid \dots \mid \Gamma_n}{\Gamma_1 \mid \dots \mid \Gamma_i, \psi, \varphi, \Gamma_i' \mid \dots \mid \Gamma_n}\,\mathrm{X}i$$

A series of weakening, contraction, and exchange inferences will often be indicated by double inference lines.

The Cut rule comes in several forms, one for every combination of distinct positions in the sequent $i \neq j$:

$$\frac{\Gamma_1 \mid \dots \mid \varphi, \Gamma_i \mid \dots \mid \Gamma_n \qquad \Delta_1 \mid \dots \mid \varphi, \Delta_j \mid \dots \mid \Delta_n}{\Gamma_1, \Delta_1 \mid \dots \mid \Gamma_n, \Delta_n}\,\mathrm{Cut}i, j$$

## seq.4 Propositional Rules for Selected Logics

mvl:seq:prl:sec The inference rules for a connective in an $n$-sided sequent calculus only depend on the characteristic truth function for the connective. Thus, if some connective is defined by the same truth function in different logics, these $n$-sided sequent rules for the connective are the same in those logics.

### Rules for ¬

The following rules for ¬ apply to Łukasiewicz and Kleene logics, and their variants.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi}{\neg\varphi, \Gamma \mid \Pi \mid \Delta}\ \neg\mathbb{F}$$

$$\frac{\Gamma \mid \varphi, \Pi \mid \Delta}{\Gamma \mid \neg\varphi, \Pi \mid \Delta}\ \neg\mathbb{U}$$

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta}{\Gamma \mid \Pi \mid \Delta, \neg\varphi}\ \neg\mathbb{T}$$

The following rules for $\neg$ apply to Gödel logic.

$$\frac{\Gamma \mid \varphi, \Pi \mid \Delta, \varphi}{\neg\varphi, \Gamma \mid \Pi \mid \Delta}\ \neg_{\mathbf{G}}\mathbb{F} \qquad \frac{\varphi, \Gamma \mid \Pi \mid \Delta}{\Gamma \mid \Pi \mid \Delta, \neg\varphi}\ \neg_{\mathbf{G}}\mathbb{T}$$

(In Gödel logic, $\neg\varphi$ can never take the value $\mathbb{U}$, so there is no rule for the middle position.)

### Rules for $\wedge$

These are the rules for $\wedge$ in Łukasiewicz, strong Kleene, and Gödel logic.

$$\frac{\varphi, \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \wedge \psi, \Gamma \mid \Pi \mid \Delta}\ \wedge\mathbb{F}$$

$$\frac{\Gamma \mid \varphi, \Pi \mid \varphi, \Delta \qquad \Gamma \mid \psi, \Pi \mid \psi, \Delta \qquad \Gamma \mid \varphi, \psi, \Pi \mid \Delta}{\Gamma \mid \varphi \wedge \psi, \Pi \mid \Delta}\ \wedge\mathbb{U}$$

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \qquad \Gamma \mid \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \wedge \psi}\ \wedge\mathbb{T}$$

### Rules for $\vee$

These are the rules for $\vee$ in Łukasiewicz, strong Kleene, and Gödel logic.

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta \qquad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \vee \psi, \Gamma \mid \Pi \mid \Delta}\ \vee\mathbb{F}$$

$$\frac{\varphi, \Gamma \mid \varphi, \Pi \mid \Delta \qquad \psi, \Gamma \mid \psi, \Pi \mid \Delta \qquad \Gamma \mid \varphi, \psi, \Pi \mid \Delta}{\Gamma \mid \varphi \lor \psi, \Pi \mid \Delta} \lor\mathbb{U}$$

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \lor \psi} \lor\mathbb{T}$$

### Rules for $\to$

These are the rules for $\to$ in Łukasiewicz logic.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \qquad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \to \psi, \Gamma \mid \Pi \mid \Delta} \to_{\mathbf{Ł_3}}\mathbb{F}$$

$$\frac{\Gamma \mid \varphi, \psi, \Pi \mid \Delta \qquad \psi, \Gamma \mid \Pi \mid \Delta, \varphi}{\Gamma \mid \varphi \to \psi, \Pi \mid \Delta} \to_{\mathbf{Ł_3}}\mathbb{U}$$

$$\frac{\varphi, \Gamma \mid \psi, \Pi \mid \Delta, \psi \qquad \varphi, \Gamma \mid \varphi, \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \to \psi} \to_{\mathbf{Ł_3}}\mathbb{T}$$

These are the rules for $\to$ in strong Kleene logic.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \qquad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \to \psi, \Gamma \mid \Pi \mid \Delta} \to_{\mathbf{Ks}}\mathbb{F}$$

$$\frac{\psi, \Gamma \mid \psi, \Pi \mid \Delta \qquad \Gamma \mid \varphi, \psi, \Pi \mid \Delta \qquad \Gamma \mid \varphi, \Pi \mid \Delta, \varphi}{\Gamma \mid \varphi \to \psi, \Pi \mid \Delta} \to_{\mathbf{Ks}}\mathbb{U}$$

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \to \psi} \to_{\mathbf{Ks}}\mathbb{T}$$

These are the rules for $\to$ in Gödel logic.

$$\frac{\Gamma \mid \varphi, \Pi \mid \Delta, \varphi \qquad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \to \psi, \Gamma \mid \Pi \mid \Delta} \to_{\mathbf{G_3}}\mathbb{F}$$

$$\frac{\Gamma \mid \psi, \Pi \mid \Delta \qquad \Gamma \mid \Pi \mid \Delta, \varphi}{\Gamma \mid \varphi \to \psi, \Pi \mid \Delta} \to_{\mathbf{G_3}}\mathbb{U}$$

$$\frac{\varphi, \Gamma \mid \psi, \Pi \mid \Delta, \psi \qquad \varphi, \Gamma \mid \varphi, \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \to \psi} \to_{\mathbf{G_3}} \mathbb{T}$$

# Photo Credits

$$
\begin{prooftree}
\AxiomC{$A \mid A \mid A$}
\RightLabel{W$\mathbb{T}$}
\UnaryInfC{$A \mid A \mid B, A$}
\RightLabel{W$\mathbb{U}$}
\UnaryInfC{$A \mid B, A \mid B, A$}
\RightLabel{W$\mathbb{U}$}
\UnaryInfC{$A \mid A, B, A \mid B, A$}
\AxiomC{$A \mid A \mid A$}
\RightLabel{W$\mathbb{T}$}
\UnaryInfC{$A \mid A \mid A, A$}
\RightLabel{W$\mathbb{T}$}
\UnaryInfC{$A \mid A \mid B, A, A$}
\RightLabel{W$\mathbb{F}$}
\UnaryInfC{$B, A \mid A \mid B, A, A$}
\RightLabel{$\to\mathbb{U}$}
\BinaryInfC{$A \mid A \to B, A \mid B, A$}
\AxiomC{$B \mid B \mid B$}
\RightLabel{W$\mathbb{U}$}
\UnaryInfC{$B \mid A, B \mid B$}
\RightLabel{X$\mathbb{U}$}
\UnaryInfC{$B \mid B, A \mid B$}
\RightLabel{W$\mathbb{U}$}
\UnaryInfC{$B \mid A, B, A \mid B$}
\RightLabel{W$\mathbb{F}$}
\UnaryInfC{$A, B \mid A, B, A \mid B$}
\RightLabel{X$\mathbb{F}$}
\UnaryInfC{$B, A \mid A, B, A \mid B$}
\AxiomC{$A \mid A \mid A$}
\RightLabel{W$\mathbb{T}$}
\UnaryInfC{$A \mid A \mid B, A$}
\RightLabel{W$\mathbb{F}$}
\UnaryInfC{$B, A \mid A \mid B, A$}
\RightLabel{W$\mathbb{F}$}
\UnaryInfC{$B, B, A \mid A \mid B, A$}
\RightLabel{$\to\mathbb{U}$}
\BinaryInfC{$B, A \mid A \to B, A \mid B$}
\RightLabel{$\to\mathbb{F}$}
\BinaryInfC{$A \to B, A \mid A \to B, A \mid B$}
\end{prooftree}
$$

Figure seq.1: Example derivation in $\mathbf{\L}_3$

# Bibliography