seq.1 Rules and Derivations

mvl:seq:rul:sec For the following, let $\Gamma, \Delta, \Pi, \Lambda$ represent finite sequences of sentences.

Definition seq.1 (Sequent). An *n-sided sequent* is an expression of the form

$$\Gamma_1 \mid \ldots \mid \Gamma_n$$

where each Γ_1 is a finite (possibly empty) sequences of sentences of the language $\mathcal{L}$.

Definition seq.2 (Initial Sequent). An *n-sided initial sequent* is an n-sided sequent of the form $\varphi \mid \ldots \mid \varphi$ for any sentence φ in the language.

If the language contains a 0-place connective $\star$, i.e., a propositional constant, then we also take the sequent $\ldots \mid \star \mid \ldots$ where $\star$ appears in the space for the truth value associated with $\widetilde{\star} \in V$, and is empty otherwise.

For each connective of an n-valued logic $\mathbf{L}$, there is a logical rule for each truth value that this connective can take in $\mathbf{L}$. Derivations in an n-sided sequent calculus for $\mathbf{L}$ are trees of sequents, where the topmost sequents are initial sequents, and if a sequent stands below one or more other sequents, it must follow correctly by a rule of inference for the connectives of $\mathbf{L}$.

Definition seq.3 (Theorems). A sentence φ is a *theorem* of an n-valued logic $\mathbf{L}$ if there is a derivation of the n-sequent containing φ in each position corresponding to a designated truth value of $\mathbf{L}$. We write $\vdash_{\mathbf{L}} \varphi$ if φ is a theorem and $\nvdash_{\mathbf{L}} \varphi$ if it is not.

Definition seq.4 (Derivability). A sentence φ is *derivable from* a set of sentences Γ in an n-valued logic $\mathbf{L}$, $\Gamma \vdash_{\mathbf{L}} \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence Γ_0' of the sentences in Γ_0 such that the following sequent has a derivation:

$$\Lambda_1 \mid \ldots \mid \Lambda_n$$

where Λ_i is φ if position i corresponds to a designated truth value, and Γ_0'otherwise. If φ is not derivable from Γ we write $\Gamma \nvdash \varphi$.

For instance, 3-valued Łukasiewicz logic has a 3-sided sequent calculus. In a 3-sided sequent $\Gamma \mid \Pi \mid \Delta$, Γ corresponds to $\mathbb{F}$, Δ to $\mathbb{T}$, and Π to $\mathbb{U}$. Axioms are $\varphi \mid \varphi \mid \varphi$. Since only $\mathbb{T}$ is designated, $\Gamma \vdash_{\mathbf{Ł}_3} \varphi$ iff the sequent $\Gamma \mid \Gamma \mid \varphi$ has a derivation. (If $\mathbb{U}$ were also designated, we would need a derivation of $\Gamma \mid \varphi \mid \varphi$.)

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Bibliography