seq.1 Propositional Rules for Selected Logics

mvl:seq:prl:sec The inference rules for a connective in an n-sided sequent calculus only depend on the characteristic truth function for the connective. Thus, if some connective is defined by the same truth function in different logics, these n-sided sequent rules for the connective are the same in those logics.

Rules for ¬

The following rules for ¬ apply to Łukasiewicz and Kleene logics, and their variants.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi}{\neg\varphi, \Gamma \mid \Pi \mid \Delta}\,\neg\mathbb{F}$$

$$\frac{\Gamma \mid \varphi, \Pi \mid \Delta}{\Gamma \mid \neg\varphi, \Pi \mid \Delta}\,\neg\mathbb{U}$$

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta}{\Gamma \mid \Pi \mid \Delta, \neg\varphi}\,\neg\mathbb{T}$$

The following rules for ¬ apply to Gödel logic.

$$\frac{\Gamma \mid \varphi, \Pi \mid \Delta, \varphi}{\neg\varphi, \Gamma \mid \Pi \mid \Delta}\,\neg_{\mathbf{G}}\mathbb{F} \qquad \frac{\varphi, \Gamma \mid \Pi \mid \Delta}{\Gamma \mid \Pi \mid \Delta, \neg\varphi}\,\neg_{\mathbf{G}}\mathbb{T}$$

(In Gödel logic, $\neg\varphi$ can never take the value $\mathbb{U}$, so there is no rule for the middle position.)

Rules for ∧

These are the rules for ∧ in Łukasiewicz, strong Kleene, and Gödel logic.

$$\frac{\varphi, \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \land \psi, \Gamma \mid \Pi \mid \Delta}\,\land\mathbb{F}$$

$$\frac{\Gamma \mid \varphi, \Pi \mid \varphi, \Delta \qquad \Gamma \mid \psi, \Pi \mid \psi, \Delta \qquad \Gamma \mid \varphi, \psi, \Pi \mid \Delta}{\Gamma \mid \varphi \land \psi, \Pi \mid \Delta}\,\land\mathbb{U}$$

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \qquad \Gamma \mid \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \land \psi}\,\land\mathbb{T}$$

Rules for $\lor$

These are the rules for $\lor$ in Łukasiewicz, strong Kleene, and Gödel logic.

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta \qquad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \lor \psi, \Gamma \mid \Pi \mid \Delta} \lor\mathbb{F}$$

$$\frac{\varphi, \Gamma \mid \varphi, \Pi \mid \Delta \qquad \psi, \Gamma \mid \psi, \Pi \mid \Delta \qquad \Gamma \mid \varphi, \psi, \Pi \mid \Delta}{\Gamma \mid \varphi \lor \psi, \Pi \mid \Delta} \lor\mathbb{U}$$

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \lor \psi} \lor\mathbb{T}$$

Rules for $\to$

These are the rules for $\to$ in Łukasiewicz logic.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \qquad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \to \psi, \Gamma \mid \Pi \mid \Delta} \to_{\mathbf{Ł_3}}\mathbb{F}$$

$$\frac{\Gamma \mid \varphi, \psi, \Pi \mid \Delta \qquad \psi, \Gamma \mid \Pi \mid \Delta, \varphi}{\Gamma \mid \varphi \to \psi, \Pi \mid \Delta} \to_{\mathbf{Ł_3}}\mathbb{U}$$

$$\frac{\varphi, \Gamma \mid \psi, \Pi \mid \Delta, \psi \qquad \varphi, \Gamma \mid \varphi, \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \to \psi} \to_{\mathbf{Ł_3}}\mathbb{T}$$

These are the rules for $\to$ in strong Kleene logic.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \qquad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \to \psi, \Gamma \mid \Pi \mid \Delta} \to_{\mathbf{Ks}}\mathbb{F}$$

$$\frac{\psi, \Gamma \mid \psi, \Pi \mid \Delta \qquad \Gamma \mid \varphi, \psi, \Pi \mid \Delta \qquad \Gamma \mid \varphi, \Pi \mid \Delta, \varphi}{\Gamma \mid \varphi \to \psi, \Pi \mid \Delta} \to_{\mathbf{Ks}}\mathbb{U}$$

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \to \psi} \to_{\mathbf{Ks}}\mathbb{T}$$

These are the rules for $\to$ in Gödel logic.

$$\frac{\mathit{\Gamma} \mid \varphi, \mathit{\Pi} \mid \mathit{\Delta}, \varphi \qquad \psi, \mathit{\Gamma} \mid \mathit{\Pi} \mid \mathit{\Delta}}{\varphi \to \psi, \mathit{\Gamma} \mid \mathit{\Pi} \mid \mathit{\Delta}} \to_{\mathbf{G}_3} \mathbb{F}$$

$$\frac{\mathit{\Gamma} \mid \psi, \mathit{\Pi} \mid \mathit{\Delta} \qquad \mathit{\Gamma} \mid \mathit{\Pi} \mid \mathit{\Delta}, \varphi}{\mathit{\Gamma} \mid \varphi \to \psi, \mathit{\Pi} \mid \mathit{\Delta}} \to_{\mathbf{G}_3} \mathbb{U}$$

$$\frac{\varphi, \mathit{\Gamma} \mid \psi, \mathit{\Pi} \mid \mathit{\Delta}, \psi \qquad \varphi, \mathit{\Gamma} \mid \varphi, \mathit{\Pi} \mid \mathit{\Delta}, \psi}{\mathit{\Gamma} \mid \mathit{\Pi} \mid \mathit{\Delta}, \varphi \to \psi} \to_{\mathbf{G}_3} \mathbb{T}$$

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Bibliography

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\begin{prooftree}
\AxiomC{$A \mid A \mid A$}
\RightLabel{W$\mathbb{T}$}
\UnaryInfC{$A \mid A \mid B, A$}
\RightLabel{W$\mathbb{U}$}
\UnaryInfC{$A \mid B, A \mid B, A$}
\RightLabel{W$\mathbb{U}$}
\UnaryInfC{$A \mid A, B, A \mid B, A$}
\AxiomC{$A \mid A \mid A$}
\RightLabel{W$\mathbb{T}$}
\UnaryInfC{$A \mid A \mid A, A$}
\RightLabel{W$\mathbb{T}$}
\UnaryInfC{$A \mid A \mid B, A, A$}
\RightLabel{W$\mathbb{F}$}
\UnaryInfC{$B, A \mid A \mid B, A, A$}
\RightLabel{$\to\mathbb{U}$}
\BinaryInfC{$A \mid A \to B, A \mid B, A$}
\AxiomC{$B \mid B \mid B$}
\RightLabel{W$\mathbb{U}$}
\UnaryInfC{$B \mid A, B \mid B$}
\RightLabel{X$\mathbb{U}$}
\UnaryInfC{$B \mid B, A \mid B$}
\RightLabel{W$\mathbb{U}$}
\UnaryInfC{$B \mid A, B, A \mid B$}
\RightLabel{W$\mathbb{F}$}
\UnaryInfC{$A, B \mid A, B, A \mid B$}
\RightLabel{X$\mathbb{F}$}
\UnaryInfC{$B, A \mid A, B, A \mid B$}
\AxiomC{$A \mid A \mid A$}
\RightLabel{W$\mathbb{T}$}
\UnaryInfC{$A \mid A \mid B, A$}
\RightLabel{W$\mathbb{F}$}
\UnaryInfC{$B, A \mid A \mid B, A$}
\RightLabel{W$\mathbb{F}$}
\UnaryInfC{$B, B, A \mid A \mid B, A$}
\RightLabel{$\to\mathbb{U}$}
\BinaryInfC{$B, A \mid A \to B, A \mid B$}
\RightLabel{$\to\mathbb{F}$}
\BinaryInfC{$A \to B, A \mid A \to B, A \mid B$}
\end{prooftree}
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Figure 1: Example derivation in $\mathbf{Ł}_3$