seq.1 Introduction

mvl:seq:int:sec

The sequent calculus for classical logic is an efficient and simple proof system. If a many-valued logic is defined by a matrix with finitely many truth values, i.e., V is finite, it is possible to provide a sequent calculus for it. The idea for how to do this comes from considering the meanings of sequents and the form of inference rules in the classical case.

Now recall that a sequent

$$\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_n$$

can be interpreted as the formula

$$(\varphi_1 \land \dots \land \varphi_m) \rightarrow (\psi_1 \lor \dots \lor \psi_n)$$

In other words, A valuation $\mathfrak{v}$ *satisfies* a sequent $\Gamma \Rightarrow \Delta$ iff either $\overline{\mathfrak{v}}(\varphi) = \mathbb{F}$ for some $\varphi \in \Gamma$ or $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ for some $\varphi \in \Delta$. On this interpretation, initial sequents $\varphi \Rightarrow \varphi$ are always satisfied, because either $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ or $\overline{}(\varphi) = \mathbb{F}$.

Here are the inference rules for the conditional in **LK**, with side formulas Γ, Δ left out:

$$\frac{\Rightarrow \varphi \qquad \psi \Rightarrow}{\varphi \rightarrow \psi \Rightarrow}\ \rightarrow\mathrm{L} \qquad\qquad \frac{\varphi \Rightarrow \psi}{\Rightarrow \varphi \rightarrow \psi}\ \rightarrow\mathrm{R}$$

If we apply the above semantic interpretation of a sequent, we can read the $\rightarrow$L rule as saying that if $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ and $\overline{\mathfrak{v}}(\psi) = \mathbb{F}$, then $\overline{\mathfrak{v}}(\varphi \rightarrow \psi) = \mathbb{F}$. Similarly, the $\rightarrow$R rule says that if either $\overline{\mathfrak{v}}(\varphi) = \mathbb{F}$ or $\overline{\mathfrak{v}}(\psi) = \mathbb{T}$, then $\overline{\mathfrak{v}}(\varphi \rightarrow \psi) = \mathbb{T}$. And in fact, these conditionals are actually biconditionals. In the case of the $\land$L and $\lor$R rules in their standard formulation, the corresponding conditionals would not be biconditionals. But there are alternative versions of these rules where they are:

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \land \psi, \Gamma \Rightarrow \Delta}\ \land\mathrm{L} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \lor \psi}\ \lor\mathrm{R}$$

This basic idea, applied to an n-valued logic, then results in a sequent calculus with n instead of two places, one for each truth value. For a three-valued logic with $V = \{\mathbb{F}, \mathbb{U}, \mathbb{T}\}$, a sequent is an expression $\Gamma \mid \Pi \mid \Delta$. It is satisfied in a valuation $\mathfrak{v}$ iff either $\overline{\mathfrak{v}}(\varphi) = \mathbb{F}$ for some $\varphi \in \Gamma$ or $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ for some $\varphi \in \Delta$ or $\overline{\mathfrak{v}}(\varphi) = \mathbb{U}$ for some $\varphi \in \Pi$. Consequently, initial sequents $\varphi \mid \varphi \mid \varphi$ are always satisfied.

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Bibliography