

Part I

Many-valued Logic

This part contains draft material on propositional many-valued logics.

Chapter 1

Syntax and Semantics

1.1 Introduction

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sec

In classical logic, we deal with **formulas** that are built from **propositional variables** using the propositional connectives \neg , \wedge , \vee , \rightarrow , and \leftrightarrow . When we define a semantics for classical logic, we do so using the two truth values \mathbb{T} and \mathbb{F} . We interpret **propositional variables** in a **valuation** \mathbf{v} , which assigns these truth values \mathbb{T} , \mathbb{F} to the **propositional variables**. Any **valuation** then determines a truth value $\bar{\mathbf{v}}(\varphi)$ for any **formula** φ , and **A formula** is satisfied in a **valuation** \mathbf{v} , $\mathbf{v} \models \varphi$, iff $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$.

Many-valued logics are generalizations of classical two-valued logic by allowing more truth values than just \mathbb{T} and \mathbb{F} . So in many-valued logic, a **valuation** \mathbf{v} is a function assigning to every **propositional variable** p one of a range of possible truth values. We'll generally call the set of allowed truth values V . Classical logic is a many-valued logic where $V = \{\mathbb{T}, \mathbb{F}\}$, and the truth value $\bar{\mathbf{v}}(\varphi)$ is computed using the familiar characteristic truth tables for the connectives.

Once we add additional truth values, we have more than one natural option for how to compute $\bar{\mathbf{v}}(\varphi)$ for the connectives we read as “and,” “or,” “not,” and “if—then.” So a many-valued logic is determined not just by the set of truth values, but also by the *truth functions* we decide to use for each connective. Once these are selected for a many-valued logic \mathbf{L} , however, the truth value $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi)$ is uniquely determined by the valuation, just like in classical logic. Many-valued logics, like classical logic, are *truth functional*.

With this semantic building blocks in hand, we can go on to define the analogs of the semantic concepts of tautology, entailment, and satisfiability. In classical logic, a **formula** is a tautology if its truth value $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ for any \mathbf{v} . In many-valued logic, we have to generalize this a bit as well. First of all, there is no requirement that the set of truth values V contains \mathbb{T} . For instance, some many-valued logics use numbers, such as all rational numbers between 0 and 1 as their set of truth values. In such a case, 1 usually plays the role of \mathbb{T} . In other logics, not just one but several truth values do. So, we require that every many-valued logic have a set V^+ of *designated values*. We can then say that

a **formula** is satisfied in a **valuation** \mathbf{v} , $\mathbf{v} \models_{\mathbf{L}} \varphi$, iff $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) \in V^+$. A **formula** φ is a tautology of the logic, $\models_{\mathbf{L}} \varphi$, iff $\bar{\mathbf{v}}(\varphi) \in V^+$ for any \mathbf{v} . And, finally, we say that φ is entailed by a set of **formulas**, $\Gamma \models_{\mathbf{L}} \varphi$, if every **valuation** that satisfies all the **formulas** in Γ also satisfies φ .

1.2 Languages and Connectives

Classical propositional logic, and many other logics, use a set supply of *propositional constants* and *connectives*. For instance, we use the following as primitives:

[mvl:syn:con:sec](#)

1. The propositional constant for **falsity** \perp .
2. The propositional constant for **truth** \top .
3. The logical connectives: \neg (negation), \wedge (conjunction), \vee (disjunction), \rightarrow (**conditional**), \leftrightarrow (**biconditional**)

The same connectives are used in many-valued logics as well. However, it is often useful to include different versions of, say, conjunction, in the same logic, and that would require different symbols to keep the versions separate. Some many-valued logics also include connectives that have no equivalent in classical logic. So, we'll be a bit more general than usual.

Definition 1.1. A *propositional language* consists of a set \mathcal{L} of *connectives*. Each connective \star has an *arity*; a connective of arity n is said to be *n-place*. Connectives of arity 0 are also called *constants*; connectives of arity 1 are called *unary*, and connectives of arity 2, *binary*.

Example 1.2. The standard language of propositional logic \mathcal{L}_0 consists of the following connectives (with associated arities): \perp (0), \neg (1), \wedge (2), \vee (2), \rightarrow (2). Most logics we consider will use this language. Some logics by tradition an convention use different symbols for some connectives. For instance, in product logic, the conjunction symbol is often \odot instead of \wedge . Sometimes it is convenient to add a new operator, e.g., the determinateness operator Δ (1-place).

1.3 Formulas

Definition 1.3 (Formula). The set $\text{Frm}(\mathcal{L})$ of *formulas* of a propositional language \mathcal{L} is defined inductively as follows:

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[mvl:syn:fml:defn:formulas](#)

1. Every **propositional variable** p_i is an atomic **formula**.
2. Every 0-place connective (propositional constant) of \mathcal{L} is an atomic **formula**.

3. If \star is an n -place connective of \mathcal{L} , and $\varphi_1, \dots, \varphi_n$ are formulas, then $\star(\varphi_1, \dots, \varphi_n)$ is a formula.
4. Nothing else is a formula.

If \star is 1-place, then $\star(\varphi_1)$ will often be written simply as $\star\varphi_1$. If \star is 2-place $\star(\varphi_1, \varphi_2)$ will often be written as $(\varphi_1 \star \varphi_2)$.

As usual, we will often silently leave out the outermost parentheses.

Example 1.4. In the standard language \mathcal{L}_0 , $p_1 \rightarrow (p_1 \wedge \neg p_2)$ is a formula. In the language of product logic, it would be written instead as $p_1 \rightarrow (p_1 \odot \neg p_2)$. If we add the 1-place Δ to the language, we would also have formulas such as $\Delta(p_1 \wedge p_2) \rightarrow (\Delta p_1 \wedge \Delta p_2)$.

1.4 Matrices

mvl:syn:mat:
sec A many-valued logic is defined by its language, its set of truth values V , a subset of designated truth values, and truth functions for its connective. Together, these elements are called a *matrix*.

mvl:syn:mat:
defn:matrix **Definition 1.5 (Matrix).** A *matrix* for the logic \mathbf{L} consists of:

1. a set of connectives making up a language \mathcal{L} ;
2. a set $V \neq \emptyset$ of truth values;
3. a set $V^+ \subseteq V$ of designated truth values;
4. for each n -place connective \star in \mathcal{L} , a truth function $\tilde{\star} : V^n \rightarrow V$. If $n = 0$, then $\tilde{\star}$ is just an element of V .

Example 1.6. The matrix for classical logic \mathbf{C} consists of:

1. The standard propositional language \mathcal{L}_0 with $\perp, \neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. For \perp , we have $\tilde{\perp} = \mathbb{F}$. The other truth functions are given by the usual truth tables (see [Figure 1.1](#)).

\simeq		$\tilde{\wedge}$	T	F	$\tilde{\vee}$	T	F	$\tilde{\supset}$	T	F
T	F	T	T	F	T	T	T	T	T	F
F	T	F	F	F	F	T	F	F	T	T

Figure 1.1: Truth functions for classical logic \mathbf{C} .

mvl:syn:mat:
fig:tf-CL

1.5 Valuations and Satisfaction

Definition 1.7 (Valuations). Let V be a set of truth values. A *valuation* for \mathcal{L} into V is a function \mathbf{v} assigning an element of V to the propositional variables of the language, i.e., $\mathbf{v}: \text{At}_0 \rightarrow V$.

mvl:syn:val:
sec

Definition 1.8. Given a valuation \mathbf{v} into the set of truth values V of a many-valued logic \mathbf{L} , define the evaluation function $\bar{\mathbf{v}}: \text{Frm}(\mathcal{L}) \rightarrow V$ inductively by:

mvl:syn:val:
defn:pValue

1. $\bar{\mathbf{v}}(\rho_n) = \mathbf{v}(\rho_n)$;
2. If \star is a 0-place connective, then $\bar{\mathbf{v}}(\star) = \tilde{\star}_{\mathbf{L}}$;
3. If \star is an n -place connective, then

$$\bar{\mathbf{v}}(\star(\varphi_1, \dots, \varphi_n)) = \tilde{\star}_{\mathbf{L}}(\bar{\mathbf{v}}(\varphi_1), \dots, \bar{\mathbf{v}}(\varphi_n)).$$

Definition 1.9 (Satisfaction). The formula φ is *satisfied* by a valuation \mathbf{v} , $\mathbf{v} \models_{\mathbf{L}} \varphi$, iff $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) \in V^+$, where V^+ is the set of designated truth values of \mathbf{L} .

mvl:syn:val:
defn:satisfaction

We write $\mathbf{v} \not\models_{\mathbf{L}} \varphi$ to mean “not $\mathbf{v} \models_{\mathbf{L}} \varphi$.” If Γ is a set of formulas, $\mathbf{v} \models_{\mathbf{L}} \Gamma$ iff $\mathbf{v} \models_{\mathbf{L}} \varphi$ for every $\varphi \in \Gamma$.

1.6 Semantic Notions

Suppose a many-valued logic \mathbf{L} is given by a matrix. Then we can define the usual semantic notions for \mathbf{L} .

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- Definition 1.10.**
1. A formula φ is *satisfiable* if for some \mathbf{v} , $\mathbf{v} \models \varphi$; it is *unsatisfiable* if for no \mathbf{v} , $\mathbf{v} \models \varphi$;
 2. A formula φ is a *tautology* if $\mathbf{v} \models \varphi$ for all valuations \mathbf{v} ;
 3. If Γ is a set of formulas, $\Gamma \models \varphi$ (“ Γ entails φ ”) if and only if $\mathbf{v} \models \varphi$ for every valuation \mathbf{v} for which $\mathbf{v} \models \Gamma$.
 4. If Γ is a set of formulas, Γ is *satisfiable* if there is a valuation \mathbf{v} for which $\mathbf{v} \models \Gamma$, and Γ is *unsatisfiable* otherwise.

We have some of the same facts for these notions as we do for the case of classical logic:

mul:syn:sem:
prop:semanticalfacts

Proposition 1.11.

1. φ is a tautology if and only if $\emptyset \models \varphi$;
2. If Γ is satisfiable then every finite subset of Γ is also satisfiable;
3. Monotony: if $\Gamma \subseteq \Delta$ and $\Gamma \models \varphi$ then also $\Delta \models \varphi$;
4. Transitivity: if $\Gamma \models \varphi$ and $\Delta \cup \{\varphi\} \models \psi$ then $\Gamma \cup \Delta \models \psi$;

mul:syn:sem:
def:Monotony

mul:syn:sem:
def:Cut

Proof. Exercise. □

Problem 1.1. Prove Proposition 1.11

In classical logic we can connect entailment and the conditional. For instance, we have the validity of *modus ponens*: If $\Gamma \models \varphi$ and $\Gamma \models \varphi \rightarrow \psi$ then $\Gamma \models \psi$. Another important relationship between \models and \rightarrow in classical logic is the semantic deduction theorem: $\Gamma \models \varphi \rightarrow \psi$ if and only if $\Gamma \cup \{\varphi\} \models \psi$. These results *do not* always hold in many-valued logics. Whether they do depends on the truth function $\widetilde{\rightarrow}$.

1.7 Many-valued logics as sublogics of C

mul:syn:sub:
sec

The usual many-valued logics are all defined using matrices in which the value of a truth-function for arguments in $\{\mathbb{T}, \mathbb{F}\}$ agrees with the classical truth functions. Specifically, in these logics, if $x \in \{\mathbb{T}, \mathbb{F}\}$, then $\widetilde{\neg}_{\mathbf{L}}(x) = \widetilde{\neg}_{\mathbf{C}}(x)$, and for \star any one of $\wedge, \vee, \rightarrow$, if $x, y \in \{\mathbb{T}, \mathbb{F}\}$, then $\widetilde{\star}_{\mathbf{L}}(x, y) = \widetilde{\star}_{\mathbf{C}}(x, y)$. In other words, the truth functions for $\neg, \wedge, \vee, \rightarrow$ restricted to $\{\mathbb{T}, \mathbb{F}\}$ are exactly the classical truth functions.

mul:syn:sub:
prop:mul-cl

Proposition 1.12. *Suppose that a many-valued logic \mathbf{L} contains the connectives $\neg, \wedge, \vee, \rightarrow$ in its language, $\mathbb{T}, \mathbb{F} \in V$, and its truth functions satisfy:*

mul:syn:sub:
prop:not

1. $\widetilde{\neg}_{\mathbf{L}}(x) = \widetilde{\neg}_{\mathbf{C}}(x)$ if $x = \mathbb{T}$ or $x = \mathbb{F}$;

mul:syn:sub:
prop:land

2. $\widetilde{\wedge}_{\mathbf{L}}(x, y) = \widetilde{\wedge}_{\mathbf{C}}(x, y)$,

mul:syn:sub:
prop:lor

3. $\widetilde{\vee}_{\mathbf{L}}(x, y) = \widetilde{\vee}_{\mathbf{C}}(x, y)$,

mul:syn:sub:
prop:lif

4. $\widetilde{\rightarrow}_{\mathbf{L}}(x, y) = \widetilde{\rightarrow}_{\mathbf{C}}(x, y)$, if $x, y \in \{\mathbb{T}, \mathbb{F}\}$.

Then, for any valuation \mathbf{v} into V such that $\mathbf{v}(p) \in \{\mathbb{T}, \mathbb{F}\}$, $\overline{\mathbf{v}}_{\mathbf{L}}(\varphi) = \overline{\mathbf{v}}_{\mathbf{C}}(\varphi)$.

Proof. By induction on φ .

1. If $\varphi \equiv p$ is atomic, we have $\overline{\mathbf{v}}_{\mathbf{L}}(\varphi) = \mathbf{v}(p) = \overline{\mathbf{v}}_{\mathbf{C}}(\varphi)$.

2. If $\varphi \equiv \neg B$, we have

$$\begin{aligned}
\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) &= \tilde{\neg}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{L}}(\psi)) && \text{by Definition 1.8} \\
&= \tilde{\neg}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi)) && \text{by inductive hypothesis} \\
&= \tilde{\neg}_{\mathbf{C}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi)) && \text{by assumption (1),} \\
&&& \text{since } \bar{\mathbf{v}}_{\mathbf{C}}(\psi) \in \{\mathbb{T}, \mathbb{F}\}, \\
&= \bar{\mathbf{v}}_{\mathbf{C}}(\varphi) && \text{by Definition 1.8.}
\end{aligned}$$

3. If $\varphi \equiv (\psi \wedge \chi)$, we have

$$\begin{aligned}
\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) &= \tilde{\wedge}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{L}}(\psi), \bar{\mathbf{v}}_{\mathbf{L}}(\chi)) && \text{by Definition 1.8} \\
&= \tilde{\wedge}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi)) && \text{by inductive hypothesis} \\
&= \tilde{\wedge}_{\mathbf{C}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi)) && \text{by assumption (2),} \\
&&& \text{since } \bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi) \in \{\mathbb{T}, \mathbb{F}\}, \\
&= \bar{\mathbf{v}}_{\mathbf{C}}(\varphi) && \text{by Definition 1.8.}
\end{aligned}$$

The cases where $\varphi \equiv (\psi \vee \chi)$ and $\varphi \equiv (\psi \rightarrow \chi)$ are similar. \square

Corollary 1.13. *If a many-valued logic satisfies the conditions of Proposition 1.12, $\mathbb{T} \in V^+$ and $\mathbb{F} \notin V^+$, then $\vDash_{\mathbf{L}} \subseteq \vDash_{\mathbf{C}}$, i.e., if $\Gamma \vDash_{\mathbf{L}} \psi$ then $\Gamma \vDash_{\mathbf{C}} \psi$. In particular, every tautology of \mathbf{L} is also a classical tautology.*

Proof. We prove the contrapositive. Suppose $\Gamma \not\vDash_{\mathbf{C}} \psi$. Then there is some valuation $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{T}, \mathbb{F}\}$ such that $\bar{\mathbf{v}}_{\mathbf{C}}(\varphi) = \mathbb{T}$ for all $\varphi \in \Gamma$ and $\bar{\mathbf{v}}_{\mathbf{C}}(\psi) = \mathbb{F}$. Since $\mathbb{T}, \mathbb{F} \in V$, the valuation \mathbf{v} is also a valuation for \mathbf{L} . By Proposition 1.12, $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \mathbb{T}$ for all $\varphi \in \Gamma$ and $\bar{\mathbf{v}}_{\mathbf{L}}(\psi) = \mathbb{F}$. Since $\mathbb{T} \in V^+$ and $\mathbb{F} \notin V^+$ that means $\mathbf{v} \vDash_{\mathbf{L}} \Gamma$ and $\mathbf{v} \not\vDash_{\mathbf{L}} \psi$, i.e., $\Gamma \not\vDash_{\mathbf{L}} \psi$. \square

Chapter 2

Three-valued Logics

2.1 Introduction

mvl:thr:int:
sec If we just add one more value \mathbb{U} to \mathbb{T} and \mathbb{F} , we get a three-valued logic. Even though there is only one more truth value, the possibilities for defining the truth-functions for \neg , \wedge , \vee , and \rightarrow are quite numerous. Then a logic might use any combination of these truth functions, and you also have a choice of making only \mathbb{T} designated, or both \mathbb{T} and \mathbb{U} .

We present here a selection of the most well-known three-valued logics, their motivations, and some of their properties.

2.2 Łukasiewicz logic

mvl:thr:luk:
sec One of the first published, worked out proposals for a many-valued logic is due to the Polish philosopher Jan Łukasiewicz in 1921. Łukasiewicz was motivated by Aristotle's sea battle problem: It seems that, *today*, the sentence "There will be a sea battle tomorrow" is neither true nor false: its truth value is not yet settled. Łukasiewicz proposed to introduce a third truth value, to such "future contingent" sentences.

I can assume without contradiction that my presence in Warsaw at a certain moment of next year, e.g., at noon on 21 December, is at the present time determined neither positively nor negatively. Hence it is possible, but not necessary, that I shall be present in Warsaw at the given time. On this assumption the proposition "I shall be in Warsaw at noon on 21 December of next year," can at the present time be neither true nor false. For if it were true now, my future presence in Warsaw would have to be necessary, which is contradictory to the assumption. If it were false now, on the other hand, my future presence in Warsaw would have to be impossible, which is also contradictory to the assumption. Therefore the proposition considered is at the moment neither true nor false and must possess a third value, different from "0" or falsity and "1" or truth.

This value we can designate by “ $\frac{1}{2}$.” It represents “the possible,” and joins “the true” and “the false” as a third value.

We will use \mathbb{U} for Łukasiewicz’s third truth value.¹

The truth functions for the connectives \neg , \wedge , and \vee are easy to determine on this interpretation: the negation of a future contingent sentence is also a future contingent sentence, so $\neg(\mathbb{U}) = \mathbb{U}$. If one conjunct of a conjunction is undetermined and the other is true, the conjunction is also undetermined—after all, depending on how the future contingent conjunct turns out, the conjunction might turn out to be true, and it might turn out to be false. So

$$\tilde{\wedge}(\mathbb{T}, \mathbb{U}) = \tilde{\wedge}(\mathbb{U}, \mathbb{T}) = \mathbb{U}.$$

If the other conjunct is false, however, it cannot turn out true, so

$$\tilde{\wedge}(\mathbb{F}, \mathbb{U}) = \tilde{\wedge}(\mathbb{U}, \mathbb{F}) = \mathbb{F}.$$

The other values (if the arguments are settled truth values, \mathbb{T} or \mathbb{F} , are like in classical logic.

For the conditional, the situation is a little trickier. Suppose q is a future contingent statement. If p is false, then $p \rightarrow q$ will be true, regardless of how q turns out, so we should set $\tilde{\rightarrow}(\mathbb{F}, \mathbb{U}) = \mathbb{T}$. And if p is true, then $q \rightarrow p$ will be true, regardless of what q turns out to be, so $\tilde{\rightarrow}(\mathbb{U}, \mathbb{T}) = \mathbb{T}$. If p is true, then $p \rightarrow q$ might turn out to be true or false, so $\tilde{\rightarrow}(\mathbb{T}, \mathbb{U}) = \mathbb{U}$. Similarly, if p is false, then $q \rightarrow p$ might turn out to be true or false, so $\tilde{\rightarrow}(\mathbb{U}, \mathbb{F}) = \mathbb{U}$. This leaves the case where p and q are both future contingents. On the basis of the motivation, we should really assign \mathbb{U} in this case. However, this would make $\varphi \rightarrow \varphi$ *not* a tautology. Łukasiewicz had not trouble giving up $\varphi \vee \neg\varphi$ and $\neg(\varphi \wedge \neg\varphi)$, but balked at giving up $\varphi \rightarrow \varphi$. So he stipulated $\tilde{\rightarrow}(\mathbb{U}, \mathbb{U}) = \mathbb{T}$.

Definition 2.1. Three-valued Łukasiewicz logic is defined using the matrix:

[mvl:thr:luk:](#)
[def:lukasiewicz](#)

1. The standard propositional language \mathcal{L}_0 with \neg , \wedge , \vee , \rightarrow .
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. Truth functions are given by the following tables:

$\tilde{\neg}$		$\tilde{\wedge}_{\mathbf{L}_3}$		\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{F}	\mathbb{F}	\mathbb{F}	\mathbb{F}

$\tilde{\vee}_{\mathbf{L}_3}$	\mathbb{T}	\mathbb{U}	\mathbb{F}	$\tilde{\rightarrow}_{\mathbf{L}_3}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{T}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{T}	\mathbb{T}	\mathbb{U}
\mathbb{F}	\mathbb{T}	\mathbb{U}	\mathbb{F}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{T}

¹Łukasiewicz here uses “possible” in a way that is uncommon today, namely to mean possible but not necessary.

As can easily be seen, any **formula** φ containing only \neg , \wedge , and \vee will take the truth value \mathbb{U} if all its **propositional variables** are assigned \mathbb{U} . So for instance, the classical tautologies $p \vee \neg p$ and $\neg(p \wedge \neg p)$ are not tautologies in \mathbf{L}_3 , since $\bar{\mathbf{v}}(\varphi) = \mathbb{U}$ whenever $\mathbf{v}(p) = \mathbb{U}$.

On **valuations** where $\mathbf{v}(p) = \mathbb{T}$ or \mathbb{F} , $\bar{\mathbf{v}}(\varphi)$ will coincide with its classical truth value.

Proposition 2.2. *If $\mathbf{v}(p) \in \{\mathbb{T}, \mathbb{F}\}$ for all p in φ , then $\bar{\mathbf{v}}_{\mathbf{L}_3}(\varphi) = \bar{\mathbf{v}}_{\mathbf{C}}(\varphi)$.*

Problem 2.1. Suppose we define $\bar{\mathbf{v}}(\varphi \leftrightarrow \psi) = \bar{\mathbf{v}}((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$ in \mathbf{L}_3 . What truth table would \leftrightarrow have?

Many classical tautologies *are* also tautologies in \mathbf{L}_3 , e.g. $\neg p \rightarrow (p \rightarrow q)$. Just like in classical logic, we can use truth tables to verify this:

p	q	\neg	p	\rightarrow	$(p$	\rightarrow	$q)$
T	T	F	T	T	T	T	T
T	U	F	T	T	T	U	U
T	F	F	T	T	T	F	F
U	T	U	U	T	U	T	T
U	U	U	U	T	U	T	U
U	F	U	U	T	U	U	F
F	T	T	F	T	F	T	T
F	U	T	F	T	F	T	U
F	F	T	F	T	F	T	F

Problem 2.2. Show that the following are tautologies in \mathbf{L}_3 :

1. $p \rightarrow (q \rightarrow p)$
2. $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
3. $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

(In (2) and (3), take $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, or refer to your solution to **Problem 2.1**.)

Problem 2.3. Show that the following classical tautologies are not tautologies in \mathbf{L}_3 :

1. $(\neg p \wedge p) \rightarrow q$
2. $((p \rightarrow q) \rightarrow p) \rightarrow p$
3. $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$

One might therefore perhaps think that although not all classical tautologies are tautologies in \mathbf{L}_3 , they should at least take either the value \mathbb{T} or the value \mathbb{U} on every **valuation**. This is not the case. A counterexample is given by

$$\neg(p \rightarrow \neg p) \vee \neg(\neg p \rightarrow p)$$

which is \mathbb{F} if p is \mathbb{U} .

Problem 2.4. Which of the following relations hold in Łukasiewicz logic? Give a truth table for each.

1. $p, p \rightarrow q \models q$
2. $\neg\neg p \models p$
3. $p \wedge q \models p$
4. $p \models p \wedge p$
5. $p \models p \vee q$

Łukasiewicz hoped to build a logic of possibility on the basis of his three-valued system, by introducing a one-place connective $\diamond\varphi$ (for “ φ is possible”) and a corresponding $\Box\varphi$ (for “ φ is necessary”):

$\tilde{\diamond}$		$\tilde{\Box}$	
T	T	T	T
U	T	U	F
F	F	F	F

In other words, p is possible iff it is not already settled as false; and p is necessary iff it is already settled as true.

Problem 2.5. Show that $\Box p \leftrightarrow \neg\diamond\neg p$ and $\diamond p \leftrightarrow \neg\Box\neg p$ are tautologies in \mathbf{L}_3 , extended with the truth tables for \Box and \diamond .

However, the shortcomings of this proposed modal logic soon became evident: However things turn out, $p \wedge \neg p$ can never turn out to be true. So even if it is not now settled (and therefore undetermined), it should count as impossible, i.e., $\neg\diamond(p \wedge \neg p)$ should be a tautology. However, if $\mathbf{v}(p) = \mathbb{U}$, then $\mathbf{v}(\neg\diamond(p \wedge \neg p)) = \mathbb{U}$. Although Łukasiewicz was correct that two truth values will not be enough to accommodate modal distinctions such as possibility and necessity, introducing a third truth value is also not enough.

2.3 Kleene logics

Stephen Kleene introduced two three-valued logics motivated by a logic in which truth values are thought of the outcomes of computational procedures: a procedure may yield T or F, but it may also fail to terminate. In that case the corresponding truth value is undefined, represented by the truth value U.

To compute the negation of a proposition φ , you would first compute the value of φ , and then return the opposite of the result. If the computation of φ does not terminate, then the entire procedure does not either: so the negation of U is U.

To compute a conjunction $\varphi \wedge \psi$, there are two options: one can first compute φ , then ψ , and then the result would be T if the outcome of both is T,

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sec

and \mathbb{F} otherwise. If either computation fails to halt, the entire procedure does as well. So in this case, the if one conjunct is undefined, the conjunction is as well. The same goes for disjunction.

However, if we can evaluate φ and ψ in parallel, we can do better. Then, if one of the two procedures halts and returns \mathbb{F} , we can stop, as the answer must be false. So in that case a conjunction with one false conjunct is false, even if the other conjunct is undefined. Similarly, when computing a disjunction in parallel, we can stop once the procedure for one of the two disjuncts has returned true: then the disjunction must be true. So in this case we can know what the outcome of a compound claim is, even if one of the components is undefined. On this interpretation, we might read \mathbb{U} as “unknown” rather than “undefined.”

The two interpretations give rise to Kleene’s strong and weak logic. The conditional is defined as equivalent to $\neg\varphi \vee \psi$.

Definition 2.3. *Strong Kleene logic \mathbf{Ks}* is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. Truth functions are given by the following tables:

\approx		$\tilde{\wedge}_{\mathbf{Ks}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{F}	\mathbb{F}	\mathbb{F}

$\tilde{\vee}_{\mathbf{Ks}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}	$\tilde{\rightarrow}_{\mathbf{Ks}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{T}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{T}	\mathbb{U}	\mathbb{U}
\mathbb{F}	\mathbb{T}	\mathbb{U}	\mathbb{F}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{T}

Definition 2.4. *Weak Kleene logic \mathbf{Kw}* is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. Truth functions are given by the following tables:

\approx		$\tilde{\wedge}_{\mathbf{Kw}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}
\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{F}	\mathbb{U}	\mathbb{F}

$\tilde{\vee}_{\mathbf{Kw}}$	T	U	F
T	T	U	T
U	U	U	U
F	T	U	F

$\tilde{\Rightarrow}_{\mathbf{Kw}}$	T	U	F
T	T	U	F
U	U	U	U
F	T	U	T

Proposition 2.5. *\mathbf{Ks} and \mathbf{Kw} have no tautologies.*

Proof. If $\mathbf{v}(p) = \mathbb{U}$ for all propositional variables p , then any formula φ will have truth value $\bar{\mathbf{v}}(\varphi) = \mathbb{U}$, since

$$\tilde{\neg}(\mathbb{U}) = \tilde{\vee}(\mathbb{U}, \mathbb{U}) = \tilde{\wedge}(\mathbb{U}, \mathbb{U}) = \tilde{\Rightarrow}(\mathbb{U}, \mathbb{U}) = \mathbb{U}$$

in both logics. As $\mathbb{U} \notin V^+$ for either \mathbf{Ks} or \mathbf{Kw} , on this valuation, φ will not be designated. □

Although both weak and strong Kleene logic have no tautologies, they have non-trivial consequence relations.

Problem 2.6. Which of the following relations hold in (a) strong and (b) weak Kleene logic? Give a truth table for each.

1. $p, p \rightarrow q \models q$
2. $p \vee q, \neg p \models q$
3. $p \wedge q \models p$
4. $p \models p \wedge p$
5. $p \models p \vee q$

Dmitry Bochvar interpreted \mathbb{U} as “meaningless” and attempted to use it to solve paradoxes such as the Liar paradox by stipulating that paradoxical sentences take the value \mathbb{U} . He introduced a logic which is essentially weak Kleene logic extended by additional connectives, two of which are “external negation” and the “is undefined” operator:

$\tilde{\sim}$		$\tilde{+}$	
T		T	
U		U	
F		F	

Problem 2.7. Can you define $\tilde{\sim}$ in Bochvar’s logic in terms of \neg and $+$, i.e., find a formula with only the propositional variable p and not involving $\tilde{\sim}$ which always takes the same truth value as $\tilde{\sim}p$? Give a truth table to show you’re right.

2.4 Gödel logics

mvl:thr:god:sec Kurt Gödel introduced a sequence of n -valued logics that each contain all formulas valid in intuitionistic logic, and are contained in classical logic. Here is the first interesting one:

mvl:thr:god:defn:goedel **Definition 2.6.** *3-valued Gödel logic* \mathbf{G} is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\perp, \neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. For \perp , we have $\tilde{\perp} = \mathbb{F}$. Truth functions for the remaining connectives are given by the following tables:

$\tilde{\neg}_{\mathbf{G}}$		$\tilde{\wedge}_{\mathbf{G}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{F}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{F}	\mathbb{F}	\mathbb{F}

$\tilde{\vee}_{\mathbf{G}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}	$\tilde{\rightarrow}_{\mathbf{G}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{T}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{T}	\mathbb{T}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{U}	\mathbb{F}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{T}

You'll notice that the truth tables for \wedge and \vee are the same as in Łukasiewicz and strong Kleene logic, but the truth tables for \neg and \rightarrow differ for each. In Gödel logic, $\tilde{\neg}(\mathbb{U}) = \mathbb{F}$. In contrast to Łukasiewicz logic and Kleene logic, $\tilde{\rightarrow}(\mathbb{U}, \mathbb{F}) = \mathbb{F}$; in contrast to Kleene logic (but as in Łukasiewicz logic), $\tilde{\rightarrow}(\mathbb{U}, \mathbb{U}) = \mathbb{T}$.

As the connection to intuitionistic logic alluded to above suggests, \mathbf{G}_3 is close to intuitionistic logic. All intuitionistic truths are tautologies in \mathbf{G}_3 , and many classical tautologies that are not valid intuitionistically also fail to be tautologies in \mathbf{G}_3 . For instance, the following are not tautologies:

$$\begin{array}{ll}
 p \vee \neg p & (p \rightarrow q) \rightarrow (\neg p \vee q) \\
 \neg \neg p \rightarrow p & \neg(p \wedge q) \rightarrow (\neg p \vee \neg q) \\
 & ((p \rightarrow q) \rightarrow p) \rightarrow p
 \end{array}$$

However, not every tautology of \mathbf{G}_3 is also intuitionistically valid, e.g., $(p \rightarrow q) \vee (q \rightarrow p)$.

Problem 2.8. Give a truth table to show that $(p \rightarrow q) \vee (q \rightarrow p)$ is a tautology of \mathbf{G}_3 .

Problem 2.9. Give truth tables that show that the following are not tautologies of \mathbf{G}_3

$$\begin{aligned}(p \rightarrow q) &\rightarrow (\neg p \vee q) \\ \neg(p \wedge q) &\rightarrow (\neg p \vee \neg q) \\ ((p \rightarrow q) \rightarrow p) &\rightarrow p\end{aligned}$$

Problem 2.10. Which of the following relations hold in Gödel logic? Give a truth table for each.

1. $p, p \rightarrow q \models q$
2. $p \vee q, \neg p \models q$
3. $p \wedge q \models p$
4. $p \models p \wedge p$
5. $p \models p \vee q$

2.5 Designating not just \mathbb{T}

So far the logics we've seen all had the set of designated truth values $V^+ = \{\mathbb{T}\}$, i.e., something counts as true iff its truth value is \mathbb{T} . But one might also count something as true if it's just not \mathbb{F} . Then one would get a logic by stipulating in the matrix, e.g., that $V^+ = \{\mathbb{T}, \mathbb{U}\}$. mvl:thr:mul:sec

Definition 2.7. The *logic of paradox* **LP** is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} and \mathbb{U} are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as in strong Kleene logic.

Definition 2.8. Halldén's *logic of nonsense* **Hal** is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$ and a 1-place connective $+$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} and \mathbb{U} are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as weak Kleene logic, plus the "is meaningless" operator:

$\tilde{\top}$	
\mathbb{T}	\mathbb{F}
\mathbb{U}	\mathbb{T}
\mathbb{F}	\mathbb{F}

By contrast to the Kleene logics with which they share truth tables, these *do* have tautologies.

mol:thr:mul:
prop:LP-taut-CL

Proposition 2.9. *The tautologies of **LP** are the same as the tautologies of classical propositional logic.*

Proof. By [Proposition 1.12](#), if $\models_{\mathbf{LP}} \varphi$ then $\models_{\mathbf{C}} \varphi$. To show the reverse, we show that if there is a valuation $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{F}, \mathbb{T}, \mathbb{U}\}$ such that $\bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$ then there is a valuation $\mathbf{v}': \text{At}_0 \rightarrow \{\mathbb{F}, \mathbb{T}\}$ such that $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{F}$. This establishes the result for **LP**, since **Ks** and **LP** have the same characteristic truth functions, and \mathbb{F} is the only truth value of **LP** that is not designated (that is the only difference between **LP** and **Ks**). Thus, if $\not\models_{\mathbf{LP}} \varphi$, for some valuation \mathbf{v} , $\bar{\mathbf{v}}_{\mathbf{LP}}(\varphi) = \bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$. By the claim we're proving, $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{F}$, i.e., $\not\models_{\mathbf{C}} \varphi$.

To establish the claim, we first define \mathbf{v}' as

$$\mathbf{v}'(p) = \begin{cases} \mathbb{T} & \text{if } \mathbf{v}(p) \in \{\mathbb{T}, \mathbb{U}\} \\ \mathbb{F} & \text{otherwise} \end{cases}$$

We now show by induction on φ that (a) if $\bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$ then $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{F}$, and (b) if $\bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{T}$ then $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{T}$

1. Induction basis: $\varphi \equiv p$. By [Definition 1.8](#), $\bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbf{v}(p) = \bar{\mathbf{v}}'_{\mathbf{C}}(\varphi)$, which implies both (a) and (b).

For the induction step, consider the cases:

2. $\varphi \equiv \neg\psi$.
 - a) Suppose $\bar{\mathbf{v}}_{\mathbf{Ks}}(\neg\psi) = \mathbb{F}$. By the definition of $\tilde{\neg}_{\mathbf{Ks}}$, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$. By inductive hypothesis, case (b), we get $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi) = \mathbb{T}$, so $\bar{\mathbf{v}}'_{\mathbf{C}}(\neg\psi) = \mathbb{F}$.
 - b) Suppose $\bar{\mathbf{v}}_{\mathbf{Ks}}(\neg\psi) = \mathbb{T}$. By the definition of $\tilde{\neg}_{\mathbf{Ks}}$, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$. By inductive hypothesis, case (a), we get $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi) = \mathbb{F}$, so $\bar{\mathbf{v}}'_{\mathbf{C}}(\neg\psi) = \mathbb{T}$.
3. $\varphi \equiv (\psi \wedge \chi)$.
 - a) Suppose $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi \wedge \chi) = \mathbb{F}$. By the definition of $\tilde{\wedge}_{\mathbf{Ks}}$, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$ or $\bar{\mathbf{v}}_{\mathbf{Ks}}(\chi) = \mathbb{F}$. By inductive hypothesis, case (a), we get $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi) = \mathbb{F}$ or $\bar{\mathbf{v}}'_{\mathbf{C}}(\chi) = \mathbb{F}$, so $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi \wedge \chi) = \mathbb{F}$.
 - b) Suppose $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi \wedge \chi) = \mathbb{T}$. By the definition of $\tilde{\wedge}_{\mathbf{Ks}}$, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$ and $\bar{\mathbf{v}}_{\mathbf{Ks}}(\chi) = \mathbb{T}$. By inductive hypothesis, case (b), we get $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi) = \mathbb{T}$ and $\bar{\mathbf{v}}'_{\mathbf{C}}(\chi) = \mathbb{T}$, so $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi \wedge \chi) = \mathbb{T}$.

The other two cases are similar, and left as exercises. Alternatively, the proof above establishes the result for all **formulas** only containing \neg and \wedge . One may now appeal to the facts that in both **Ks** and **C**, for any \mathbf{v} , $\bar{\mathbf{v}}(\psi \vee \chi) = \bar{\mathbf{v}}(\neg(\neg\psi \wedge \neg\chi))$ and $\bar{\mathbf{v}}(\psi \rightarrow \chi) = \bar{\mathbf{v}}(\neg(\psi \wedge \neg\chi))$. \square

Problem 2.11. Complete the proof **Proposition 2.9**, i.e., establish (a) and (b) for the cases where $\varphi \equiv (\psi \vee \chi)$ and $\varphi \equiv (\psi \rightarrow \chi)$.

Problem 2.12. Prove that every classical tautology is a tautology in **Hal**.

Although they have the same tautologies as classical logic, their consequence relations are different. **LP**, for instance, is *paraconsistent* in that $\neg p, p \not\vdash q$, and so the principle of explosion $\neg\varphi, \varphi \vdash \psi$ does not hold in general. (It holds for some cases of φ and ψ , e.g., if ψ is a tautology.)

Problem 2.13. Which of the following relations hold in (a) **LP** and in (b) **Hal**? Give a truth table for each.

1. $p, p \rightarrow q \vDash q$
2. $\neg q, p \rightarrow q \vDash \neg p$
3. $p \vee q, \neg p \vDash q$
4. $\neg p, p \vDash q$
5. $p \vDash p \vee q$
6. $p \rightarrow q, q \rightarrow r \vDash p \rightarrow r$

What if you make \mathbb{U} designated in **L₃**?

Definition 2.10. The logic *3-valued R-Mingle* **RM₃** is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\perp, \neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} and \mathbb{U} are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as Łukasiewicz logic **L₃**.

Problem 2.14. Which of the following relations hold in **RM₃**?

1. $p, p \rightarrow q \vDash q$
2. $p \vee q, \neg p \vDash q$
3. $\neg p, p \vDash q$
4. $p \vDash p \vee q$

Different truth tables can sometimes generate the same logic (entailment relation) just by changing the designated values. E.g., this happens if in Gödel logic we take $V^+ = \{\mathbb{T}, \mathbb{U}\}$ instead of $\{\mathbb{T}\}$.

mol:thr:mul: **Proposition 2.11.** *prop:gl-udes* The matrix with $V = \{\mathbb{F}, \mathbb{U}, \mathbb{T}\}$, $V^+ = \{\mathbb{T}, \mathbb{U}\}$, and the truth functions of 3-valued Gödel logic defines classical logic.

Proof. Exercise. □

Problem 2.15. Prove **Proposition 2.11** by showing that for the logic \mathbf{L} defined just like Gödel logic but with $V^+ = \{\mathbb{T}, \mathbb{U}\}$, if $\Gamma \not\vdash_{\mathbf{L}} \psi$ then $\Gamma \not\vdash_{\mathbf{C}} \psi$. Use the ideas of **Proposition 2.9**, except instead of proving properties (a) and (b), show that $\bar{\mathbf{v}}_{\mathbf{G}}(\varphi) = \mathbb{F}$ iff $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{F}$ (and hence that $\bar{\mathbf{v}}_{\mathbf{G}}(\varphi) \in \{\mathbb{T}, \mathbb{U}\}$ iff $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{T}$). Explain why this establishes the proposition.

Chapter 3

Infinite-valued Logics

3.1 Introduction

The number of truth values of a matrix need not be finite. An obvious choice for a set of infinitely many truth values is the set of rational numbers between 0 and 1, $V_\infty = [0, 1] \cap \mathbb{Q}$, i.e., mvl:inf:int:
sec

$$V_\infty = \left\{ \frac{n}{m} : n, m \in \mathbb{N} \text{ and } n \leq m \right\}.$$

When considering this infinite truth value set, it is often useful to also consider the subsets

$$V_m = \left\{ \frac{n}{m-1} : n \in \mathbb{N} \text{ and } n \leq m \right\}$$

For instance, V_5 is the set with 5 evenly spaced truth values,

$$V_5 = \left\{ 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1 \right\}.$$

In logics based on these truth value sets, usually only 1 is designated, i.e., $V^+ = \{1\}$. In other words, we let 1 play the role of (absolute) truth, 0 as absolute falsity, but **formulas** may take any intermediate value in V .

One can also consider the set $V_{[0,1]} = [0, 1]$ of all *real* numbers between 0 and 1, or other infinite subsets of $[0, 1]$, however. Logics with this truth value set are often called *fuzzy*.

3.2 Łukasiewicz logic

mvl:inf:luk:
sec

This is a short “stub” of a section on infinite-valued Łukasiewicz logic.

[mvl:inf:luk:](#)
[def:lukasiewicz](#)

Definition 3.1. Infinite-valued Łukasiewicz logic \mathbf{L}_∞ is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$.
2. The set of truth values V_∞ .
3. 1 is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:

$$\begin{aligned}\tilde{\neg}_{\mathbf{L}}(x) &= 1 - x \\ \tilde{\wedge}_{\mathbf{L}}(x, y) &= \min(x, y) \\ \tilde{\vee}_{\mathbf{L}}(x, y) &= \max(x, y) \\ \tilde{\rightarrow}_{\mathbf{L}}(x, y) &= \min(1, 1 - (x - y)) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - (x - y) & \text{otherwise.} \end{cases}\end{aligned}$$

m -valued Łukasiewicz logic is defined the same, except $V = V_m$.

Proposition 3.2. The logic \mathbf{L}_3 defined by [Definition 2.1](#) is the same as \mathbf{L}_3 defined by [Definition 3.1](#).

Proof. This can be seen by comparing the truth tables for the connectives given in [Definition 2.1](#) with the truth tables determined by the equations in [Definition 3.1](#):

$\tilde{\neg}$		$\tilde{\wedge}_{\mathbf{L}_3}$	1	1/2	0	
1	0	1	1	1/2	0	
1/2	1/2	1/2	1/2	1/2	0	
0	1	0	0	0	0	

$\tilde{\vee}_{\mathbf{L}_3}$	1	1/2	0	$\tilde{\rightarrow}_{\mathbf{L}_3}$	1	1/2	0	
1	1	1	1	1	1	1/2	0	□
1/2	1	1/2	1/2	1/2	1	1	1/2	
0	1	1/2	0	0	1	1	1	

[mvl:inf:luk:](#)
[prop:luk-infty-m](#)

Proposition 3.3. If $\Gamma \models_{\mathbf{L}_\infty} \psi$ then $\Gamma \models_{\mathbf{L}_m} \psi$ for all $m \geq 2$.

Proof. Exercise. □

Problem 3.1. Prove [Proposition 3.3](#).

In fact, the converse holds as well.

Infinite-valued Łukasiewicz logic is the most popular fuzzy logic. In the fuzzy logic literature, the conditional is often defined as $\neg\varphi \vee \psi$. The result would be an infinite-valued strong Kleene logic.

Problem 3.2. Show that $(p \rightarrow q) \vee (q \rightarrow p)$ is a tautology of \mathbf{L}_∞ .

3.3 Gödel logics

mv:inf:god:
sec

This is a short “stub” of a section on infinite-valued Gödel logic.

Definition 3.4. Infinite-valued Gödel logic \mathbf{G}_∞ is defined using the matrix:

mv:inf:god:
def:goedel

1. The standard propositional language \mathcal{L}_0 with $\perp, \neg, \wedge, \vee, \rightarrow$.
2. The set of truth values V_∞ .
3. 1 is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:

$$\begin{aligned} \tilde{\perp} &= 0 \\ \tilde{\neg}_{\mathbf{G}}(x) &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \\ \tilde{\wedge}_{\mathbf{G}}(x, y) &= \min(x, y) \\ \tilde{\vee}_{\mathbf{G}}(x, y) &= \max(x, y) \\ \tilde{\rightarrow}_{\mathbf{G}}(x, y) &= \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases} \end{aligned}$$

m -valued Gödel logic is defined the same, except $V = V_m$.

Proposition 3.5. *The logic \mathbf{G}_3 defined by Definition 2.6 is the same as \mathbf{G}_3 defined by Definition 3.4.*

Proof. This can be seen by comparing the truth tables for the connectives given in Definition 2.6 with the truth tables determined by the equations in Definition 3.4:

$\tilde{\neg}_{\mathbf{G}_3}$		$\tilde{\wedge}_{\mathbf{G}}$	1	1/2	0
1	0	1	1	1/2	0
1/2	0	1/2	1/2	1/2	0
0	1	0	0	0	0

$\tilde{\vee}_{\mathbf{G}}$	1	1/2	0	$\tilde{\rightarrow}_{\mathbf{G}}$	1	1/2	0
1	1	1	1	1	1	1/2	0
1/2	1	1/2	1/2	1/2	1	1	0
0	1	1/2	0	0	1	1	1

□

Proposition 3.6. *If $\Gamma \vDash_{\mathbf{G}_\infty} \psi$ then $\Gamma \vDash_{\mathbf{G}_m} \psi$ for all $m \geq 2$.*

mv:inf:god:
prop:god-infity-m

Proof. Exercise. □

Problem 3.3. Prove [Proposition 3.6](#).

In fact, the converse holds as well.

Like \mathbf{G}_3 , \mathbf{G}_∞ has all intuitionistically valid formulas as tautologies, and the same examples of non-tautologies are non-tautologies of \mathbf{G}_∞ :

$$\begin{array}{ll} p \vee \neg p & (p \rightarrow q) \rightarrow (\neg p \vee q) \\ \neg\neg p \rightarrow p & \neg(p \wedge q) \rightarrow (\neg p \vee \neg q) \\ & ((p \rightarrow q) \rightarrow p) \rightarrow p \end{array}$$

The example of an intuitionistically invalid formula that is nevertheless a tautology of \mathbf{G}_3 , $(p \rightarrow q) \vee (q \rightarrow p)$, is also a tautology in \mathbf{G}_∞ . In fact, \mathbf{G}_∞ can be characterized as intuitionistic logic to which the schema $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ is added. This was shown by Michael Dummett, and so \mathbf{G}_∞ is often referred to as Gödel-Dummett logic **LC**.

Problem 3.4. Show that $(p \rightarrow q) \vee (q \rightarrow p)$ is a tautology of \mathbf{G}_∞ .

Chapter 4

Sequent Calculus

4.1 Introduction

The sequent calculus for classical logic is an efficient and simple proof system. If a many-valued logic is defined by a matrix with finitely many truth values, i.e., V is finite, it is possible to provide a sequent calculus for it. The idea for how to do this comes from considering the meanings of sequents and the form of inference rules in the classical case.

mvl:seq:int:
sec

Now recall that a sequent

$$\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_n$$

can be interpreted as the formula

$$(\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow (\psi_1 \vee \dots \vee \psi_n)$$

In other words, A valuation \mathbf{v} satisfies a sequent $\Gamma \Rightarrow \Delta$ iff either $\bar{\mathbf{v}}(\varphi) = \mathbb{F}$ for some $\varphi \in \Gamma$ or $\bar{\mathbf{v}}(\psi) = \mathbb{T}$ for some $\psi \in \Delta$. On this interpretation, initial sequents $\varphi \Rightarrow \varphi$ are always satisfied, because either $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ or $\bar{\mathbf{v}}(\varphi) = \mathbb{F}$.

Here are the inference rules for the conditional in **LK**, with side formulas Γ, Δ left out:

$\frac{\Rightarrow \varphi \quad \psi \Rightarrow}{\varphi \rightarrow \psi \Rightarrow} \rightarrow\text{L}$	$\frac{\varphi \Rightarrow \psi}{\Rightarrow \varphi \rightarrow \psi} \rightarrow\text{R}$
---	---

If we apply the above semantic interpretation of a sequent, we can read the $\rightarrow\text{L}$ rule as saying that if $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ and $\bar{\mathbf{v}}(\psi) = \mathbb{F}$, then $\bar{\mathbf{v}}(\varphi \rightarrow \psi) = \mathbb{F}$. Similarly, the $\rightarrow\text{R}$ rule says that if either $\bar{\mathbf{v}}(\varphi) = \mathbb{F}$ or $\bar{\mathbf{v}}(\psi) = \mathbb{T}$, then $\bar{\mathbf{v}}(\varphi \rightarrow \psi) = \mathbb{T}$. And in fact, these conditionals are actually biconditionals. In the case of the $\wedge\text{L}$ and $\vee\text{R}$ rules in their standard formulation, the corresponding conditionals would not be biconditionals. But there are alternative versions of these rules where they are:

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge L \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee R$$

This basic idea, applied to an n -valued logic, then results in a sequent calculus with n instead of two places, one for each truth value. For a three-valued logic with $V = \{\mathbb{F}, \mathbb{U}, \mathbb{T}\}$, a sequent is an expression $\Gamma \mid \Pi \mid \Delta$. It is satisfied in a **valuation** \mathbf{v} iff either $\bar{\mathbf{v}}(\varphi) = \mathbb{F}$ for some $\varphi \in \Gamma$ or $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ for some $\varphi \in \Delta$ or $\bar{\mathbf{v}}(\varphi) = \mathbb{U}$ for some $\varphi \in \Pi$. Consequently, initial sequents $\varphi \mid \varphi \mid \varphi$ are always satisfied.

4.2 Rules and Derivations

mvl:seq:rul:
sec For the following, let $\Gamma, \Delta, \Pi, \Lambda$ represent finite sequences of **sentences**.

Definition 4.1 (Sequent). An n -sided sequent is an expression of the form

$$\Gamma_1 \mid \dots \mid \Gamma_n$$

where each Γ_i is a finite (possibly empty) sequences of **sentences** of the language \mathcal{L} .

Definition 4.2 (Initial Sequent). An n -sided *initial sequent* is an n -sided sequent of the form $\varphi \mid \dots \mid \varphi$ for any **sentence** φ in the language.

If the language contains a 0-place connective \star , i.e., a propositional constant, then we also take the sequent $\dots \mid \star \mid \dots$ where \star appears in the space for the truth value associated with $\bar{\mathbf{x}} \in V$, and is empty otherwise.

For each connective of an n -valued logic \mathbf{L} , there is a logical rule for each truth value that this connective can take in \mathbf{L} . **Derivations** in an n -sided sequent calculus for \mathbf{L} are trees of sequents, where the topmost sequents are initial sequents, and if a sequent stands below one or more other sequents, it must follow correctly by a rule of inference for the connectives of \mathbf{L} .

Definition 4.3 (Theorems). A **sentence** φ is a *theorem* of an n -valued logic \mathbf{L} if there is a **derivation** of the n -sequent containing φ in each position corresponding to a designated truth value of \mathbf{L} . We write $\vdash_{\mathbf{L}} \varphi$ if φ is a theorem and $\not\vdash_{\mathbf{L}} \varphi$ if it is not.

Definition 4.4 (Derivability). A **sentence** φ is *derivable from* a set of **sentences** Γ in an n -valued logic \mathbf{L} , $\Gamma \vdash_{\mathbf{L}} \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence Γ'_0 of the **sentences** in Γ_0 such that the following sequent has a **derivation**:

$$\Lambda_1 \mid \dots \mid \Lambda_n$$

where Λ_i is φ if position i corresponds to a designated truth value, and Γ'_0 otherwise. If φ is not **derivable** from Γ we write $\Gamma \not\vdash \varphi$.

For instance, 3-valued Łukasiewicz logic has a 3-sided sequent calculus. In a 3-sided sequent $\Gamma \mid \Pi \mid \Delta$, Γ corresponds to \mathbb{F} , Δ to \mathbb{T} , and Π to \mathbb{U} . Axioms are $\varphi \mid \varphi \mid \varphi$. Since only \mathbb{T} is designated, $\Gamma \vdash_{\mathbf{L}_3} \varphi$ iff the sequent $\Gamma \mid \Gamma \mid \varphi$ has a **derivation**. (If \mathbb{U} were also designated, we would need a **derivation** of $\Gamma \mid \varphi \mid \varphi$.)

4.3 Structural Rules

The structural rules for n -sided sequent calculus operate as in the classical case, mvl:seq:str:sec except for each position i .

$$\frac{\Gamma_1 \mid \dots \mid \Gamma_i \mid \dots \mid \Gamma_n}{\Gamma_1 \mid \dots \mid \varphi, \Gamma_i \mid \dots \mid \Gamma_n} \text{Wi}$$

$$\frac{\Gamma_1 \mid \dots \mid \varphi, \varphi, \Gamma_i \mid \dots \mid \Gamma_n}{\Gamma_1 \mid \dots \mid \varphi, \Gamma_i \mid \dots \mid \Gamma_n} \text{Ci}$$

$$\frac{\Gamma_1 \mid \dots \mid \Gamma_i, \varphi, \psi, \Gamma'_i \mid \dots \mid \Gamma_n}{\Gamma_1 \mid \dots \mid \Gamma_i, \psi, \varphi, \Gamma'_i \mid \dots \mid \Gamma_n} \text{Xi}$$

A series of weakening, contraction, and exchange inferences will often be indicated by double inference lines.

The Cut rule comes in several forms, one for every combination of distinct positions in the sequent $i \neq j$:

$$\frac{\Gamma_1 \mid \dots \mid \varphi, \Gamma_i \mid \dots \mid \Gamma_n \quad \Delta_1 \mid \dots \mid \varphi, \Delta_j \mid \dots \mid \Delta_n}{\Gamma_1, \Delta_1 \mid \dots \mid \Gamma_n, \Delta_n} \text{Cut}_{i,j}$$

4.4 Propositional Rules for Selected Logics

The inference rules for a connective in an n -sided sequent calculus only depend mvl:seq:prl:sec on the characteristic truth function for the connective. Thus, if some connective is defined by the same truth function in different logics, these n -sided sequent rules for the connective are the same in those logics.

Rules for \neg

The following rules for \neg apply to Łukasiewicz and Kleene logics, and their variants.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi}{\neg\varphi, \Gamma \mid \Pi \mid \Delta} \neg\mathbb{F}$$

$$\frac{\Gamma \mid \varphi, \Pi \mid \Delta}{\Gamma \mid \neg\varphi, \Pi \mid \Delta} \neg\mathbb{U}$$

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta}{\Gamma \mid \Pi \mid \Delta, \neg\varphi} \neg\mathbb{T}$$

The following rules for \neg apply to Gödel logic.

$$\frac{\Gamma \mid \varphi, \Pi \mid \Delta, \varphi}{\neg\varphi, \Gamma \mid \Pi \mid \Delta} \neg\mathbf{G}\mathbb{F} \qquad \frac{\varphi, \Gamma \mid \Pi \mid \Delta}{\Gamma \mid \Pi \mid \Delta, \neg\varphi} \neg\mathbf{G}\mathbb{T}$$

(In Gödel logic, $\neg\varphi$ can never take the value \mathbb{U} , so there is no rule for the middle position.)

Rules for \wedge

These are the rules for \wedge in Łukasiewicz, strong Kleene, and Gödel logic.

$$\frac{\varphi, \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \wedge \psi, \Gamma \mid \Pi \mid \Delta} \wedge\mathbb{F}$$

$$\frac{\Gamma \mid \varphi, \Pi \mid \varphi, \Delta \quad \Gamma \mid \psi, \Pi \mid \psi, \Delta \quad \Gamma \mid \varphi, \psi, \Pi \mid \Delta}{\Gamma \mid \varphi \wedge \psi, \Pi \mid \Delta} \wedge\mathbb{U}$$

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \quad \Gamma \mid \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \wedge \psi} \wedge\mathbb{T}$$

Rules for \vee

These are the rules for \vee in Łukasiewicz, strong Kleene, and Gödel logic.

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta \quad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \vee \psi, \Gamma \mid \Pi \mid \Delta} \vee\mathbb{F}$$

$$\frac{\varphi, \Gamma \mid \varphi, \Pi \mid \Delta \quad \psi, \Gamma \mid \psi, \Pi \mid \Delta \quad \Gamma \mid \varphi, \psi, \Pi \mid \Delta}{\Gamma \mid \varphi \vee \psi, \Pi \mid \Delta} \vee \text{U}$$

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \vee \psi} \vee \text{T}$$

Rules for \rightarrow

These are the rules for \rightarrow in Łukasiewicz logic.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \quad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \rightarrow \psi, \Gamma \mid \Pi \mid \Delta} \rightarrow_{\mathbf{L}_3} \text{F}$$

$$\frac{\Gamma \mid \varphi, \psi, \Pi \mid \Delta \quad \psi, \Gamma \mid \Pi \mid \Delta, \varphi}{\Gamma \mid \varphi \rightarrow \psi, \Pi \mid \Delta} \rightarrow_{\mathbf{L}_3} \text{U}$$

$$\frac{\varphi, \Gamma \mid \psi, \Pi \mid \Delta, \psi \quad \varphi, \Gamma \mid \varphi, \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \rightarrow \psi} \rightarrow_{\mathbf{L}_3} \text{T}$$

These are the rules for \rightarrow in strong Kleene logic.

$$\frac{\Gamma \mid \Pi \mid \Delta, \varphi \quad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \rightarrow \psi, \Gamma \mid \Pi \mid \Delta} \rightarrow_{\mathbf{K}_s} \text{F}$$

$$\frac{\psi, \Gamma \mid \psi, \Pi \mid \Delta \quad \Gamma \mid \varphi, \psi, \Pi \mid \Delta \quad \Gamma \mid \varphi, \Pi \mid \Delta, \varphi}{\Gamma \mid \varphi \rightarrow \psi, \Pi \mid \Delta} \rightarrow_{\mathbf{K}_s} \text{U}$$

$$\frac{\varphi, \Gamma \mid \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \rightarrow \psi} \rightarrow_{\mathbf{K}_s} \text{T}$$

These are the rules for \rightarrow in Gödel logic.

$$\frac{\Gamma \mid \varphi, \Pi \mid \Delta, \varphi \quad \psi, \Gamma \mid \Pi \mid \Delta}{\varphi \rightarrow \psi, \Gamma \mid \Pi \mid \Delta} \rightarrow_{\mathbf{G}_3} \text{F}$$

$$\frac{\Gamma \mid \psi, \Pi \mid \Delta \quad \Gamma \mid \Pi \mid \Delta, \varphi}{\Gamma \mid \varphi \rightarrow \psi, \Pi \mid \Delta} \rightarrow_{\mathbf{G}_3} \text{U}$$

$$\frac{\varphi, \Gamma \mid \psi, \Pi \mid \Delta, \psi \quad \varphi, \Gamma \mid \varphi, \Pi \mid \Delta, \psi}{\Gamma \mid \Pi \mid \Delta, \varphi \rightarrow \psi} \rightarrow_{\mathbf{G}_3} \mathbb{T}$$

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Bibliography