Chapter udf

Infinite-valued Logics

inf.1 Introduction

mvl:inf:int:sec The number of truth values of a matrix need not be finite. An obvious choice for a set of infinitely many truth values is the set of rational numbers between 0 and 1, $V_\infty = [0, 1] \cap \mathbb{Q}$, i.e.,

$$V_\infty = \{\frac{n}{m} : n, m \in \mathbb{N} \text{ and } n \leq m\}.$$

When considering this infinite truth value set, it is often useful to also consider the subsets

$$V_m = \{\frac{n}{m-1} : n \in \mathbb{N} \text{ and } n \leq m\}$$

For instance, V_5 is the set with 5 evenly spaced truth values,

$$V_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}.$$

In logics based on these truth value sets, usually only 1 is designated, i.e., $V^+ = \{1\}$. In other words, we let 1 play the role of (absolute) truth, 0 as absolute falsity, but formulas may take any intermediate value in V.

One can also consider the set $V_{[0,1]} = [0, 1]$ of all *real* numbers between 0 and 1, or other infinite subsets of $[0, 1]$, however. Logics with this truth value set are often called *fuzzy*.

inf.2 Łukasiewicz logic

mvl:inf:luk:sec

This is a short "stub" of a section on infinite-valued Łukasiewicz logic.

Definition inf.1. Infinite-valued Łukasiewicz logic $\mathbf{Ł}_\infty$ is defined using the matrix: mvl:inf:luk: def:lukasiewicz

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\wedge$, $\vee$, $\rightarrow$.
2. The set of truth values V_∞.
3. 1 is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:

$$\begin{aligned}
\widetilde{\neg}_{\mathbf{Ł}}(x) &= 1 - x \\
\widetilde{\wedge}_{\mathbf{Ł}}(x, y) &= \min(x, y) \\
\widetilde{\vee}_{\mathbf{Ł}}(x, y) &= \max(x, y) \\
\widetilde{\rightarrow}_{\mathbf{Ł}}(x, y) &= \min(1, 1 - (x - y)) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - (x - y) & \text{otherwise.} \end{cases}
\end{aligned}$$

m-valued Łukasiewicz logic is defined the same, except $V = V_m$.

Proposition inf.2. *The logic $\mathbf{Ł}_3$ defined by* **??** *is the same as $\mathbf{Ł}_3$ defined by Definition inf.1.*

Proof. This can be seen by comparing the truth tables for the connectives given in **??** with the truth tables determined by the equations in Definition inf.1:

$\widetilde{\neg}$	
1	0
1/2	1/2
0	1

$\widetilde{\wedge}_{\mathbf{Ł}_3}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\widetilde{\vee}_{\mathbf{Ł}_3}$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\widetilde{\rightarrow}_{\mathbf{Ł}_3}$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

□

Proposition inf.3. *If $\Gamma \vDash_{\mathbf{Ł}_\infty} \psi$ then $\Gamma \vDash_{\mathbf{Ł}_m} \psi$ for all $m \geq 2$.* *mvl:inf:luk: prop:luk-infty-m*

Proof. Exercise. □

Problem inf.1. Prove Proposition inf.3.

In fact, the converse holds as well.

Infinite-valued Łukasiewicz logic is the most popular fuzzy logic. In the fuzzy logic literature, the conditional is often defined as $\neg\varphi \vee \psi$. The result would be an infinite-valued strong Kleene logic.

Problem inf.2. Show that $(p \rightarrow q) \vee (q \rightarrow p)$ is a tautology of $\mathbf{Ł}_\infty$.

inf.3 Gödel logics

mvl:inf:god:sec

This is a short "stub" of a section on infinite-valued Gödel logic.

mvl:inf:god:def:goedel **Definition inf.4.** Infinite-valued Gödel logic $\mathbf{G}_\infty$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot$, $\neg$, $\wedge$, $\vee$, $\rightarrow$.
2. The set of truth values V_∞.
3. 1 is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:

$$\widetilde{\bot} = 0$$
$$\widetilde{\neg}_{\mathbf{G}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\widetilde{\wedge}_{\mathbf{G}}(x, y) = \min(x, y)$$
$$\widetilde{\vee}_{\mathbf{G}}(x, y) = \max(x, y)$$
$$\widetilde{\rightarrow}_{\mathbf{G}}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$

m-valued Gödel logic is defined the same, except $V = V_m$.

Proposition inf.5. *The logic $\mathbf{G}_3$ defined by* **??** *is the same as $\mathbf{G}_3$ defined by Definition inf.4.*

Proof. This can be seen by comparing the truth tables for the connectives given in **??** with the truth tables determined by the equations in Definition inf.4:

$\widetilde{\neg}_{\mathbf{G}_3}$	
1	0
1/2	0
0	1

$\widetilde{\wedge}_{\mathbf{G}}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\widetilde{\vee}_{\mathbf{G}}$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\widetilde{\rightarrow}_{\mathbf{G}}$	1	1/2	0
1	1	1/2	0
1/2	1	1	0
0	1	1	1

□

mvl:inf:god:prop:god-infty-m **Proposition inf.6.** *If $\Gamma \vDash_{\mathbf{G}_\infty} \psi$ then $\Gamma \vDash_{\mathbf{G}_m} \psi$ for all $m \geq 2$.*

Proof. Exercise. □

Problem inf.3. Prove Proposition inf.6.

In fact, the converse holds as well.

Like $\mathbf{G}_3$, $\mathbf{G}_\infty$ has all intuitionistically valid formulas as tautologies, and the same examples of non-tautologies are non-tautologies of $\mathbf{G}_\infty$:

$$\begin{array}{ll} p \lor \neg p & (p \to q) \to (\neg p \lor q) \\ \neg\neg p \to p & \neg(\neg p \land \neg q) \to (p \lor q) \\ ((p \to q) \to p) \to p & \neg(p \to q) \to (p \land \neg q) \end{array}$$

The example of an intuitionistically invalid formula that is nevertheless a tautology of $\mathbf{G}_3$, $(p \to q) \lor (q \to p)$, is also a tautology in $\mathbf{G}_\infty$. In fact, $\mathbf{G}_\infty$ can be characterized as intuitionistic logic to which the schema $(\varphi \to \psi) \lor (\psi \to \varphi)$ is added. This was shown by Michael Dummett, and so $\mathbf{G}_\infty$ is often referred to as Gödel–Dummett logic **LC**.

Problem inf.4. Show that $(p \to q) \lor (q \to p)$ is a tautology of $\mathbf{G}_\infty$.

Problem inf.5. Show that $(p \to q) \lor (q \to r) \lor (r \to s)$, which is a tautology of $\mathbf{G}_3$, is not a tautology of $\mathbf{G}_\infty$.

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Bibliography