**inf.1 Gödel logics**

This is a short “stub” of a section on infinite-valued Gödel logic.

**Definition inf.1.** Infinite-valued Gödel logic $G_{\infty}$ is defined using the matrix:

1. The standard propositional language $L_0$ with $\bot, \neg, \land, \lor, \rightarrow$.
2. The set of truth values $V_{\infty}$.
3. $1$ is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:
   
   \[
   \begin{align*}
   \neg_G(x) &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \\
   \land_G(x, y) &= \min(x, y) \\
   \lor_G(x, y) &= \max(x, y) \\
   \rightarrow_G(x, y) &= \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}
   \end{align*}
   \]

$m$-valued Gödel logic is defined the same, except $V = V_m$.

**Proposition inf.2.** The logic $G_3$ defined by ?? is the same as $G_3$ defined by Definition inf.1.

**Proof.** This can be seen by comparing the truth tables for the connectives given in ?? with the truth tables determined by the equations in Definition inf.1:

\[
\begin{array}{c|c|c|c}
\neg_G & \land_G & \lor_G \\
\hline
1 & 0 & 1 \\
1/2 & 0 & 1/2 \\
0 & 1 & 0 \\
\end{array}
\begin{array}{c|c|c|c}
\rightarrow_G & \leftarrow_G \\
\hline
1 & 1/2 & 0 \\
1/2 & 1/2 & 1/2 \\
0 & 1 & 0 \\
\end{array}
\]

**Proposition inf.3.** If $\Gamma \vdash_{G_{\infty}} \psi$ then $\Gamma \vdash_{G_m} \psi$ for all $m \geq 2$.

**Proof.** Exercise.
Problem inf.1. Prove Proposition inf.3.

In fact, the converse holds as well.

Like $G_3$, $G_\infty$ has all intuitionistically valid formulas as tautologies, and the same examples of non-tautologies are non-tautologies of $G_\infty$:

\[
\begin{align*}
 p \lor \lnot p & \quad (p \to q) \to (\lnot p \lor q) \\
 \lnot \lnot p \to p & \quad \lnot (p \land q) \to (\lnot p \lor \lnot q) \\
 ((p \to q) \to p) \to p &
\end{align*}
\]

The example of an intuitionistically invalid formula that is nevertheless a tautology of $G_3$, $(p \to q) \lor (q \to p)$, is also a tautology in $G_\infty$. In fact, $G_\infty$ can be characterized as intuitionistic logic to which the schema $(\varphi \to \psi) \lor (\psi \to \varphi)$ is added. This was shown by Michael Dummett, and so $G_\infty$ is often referred to as Gödel-Dummett logic LC.

Problem inf.2. Show that $(p \to q) \lor (q \to p)$ is a tautology of $G_\infty$.

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Bibliography