

inf.1 Gödel logics

mvl:inf:god:
sec

This is a short “stub” of a section on infinite-valued Gödel logic.

mvl:inf:god:
def:goedel

Definition inf.1. Infinite-valued Gödel logic \mathbf{G}_∞ is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\perp, \neg, \wedge, \vee, \rightarrow$.
2. The set of truth values V_∞ .
3. 1 is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:

$$\begin{aligned}\tilde{\perp} &= 0 \\ \tilde{\neg}_{\mathbf{G}}(x) &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \\ \tilde{\wedge}_{\mathbf{G}}(x, y) &= \min(x, y) \\ \tilde{\vee}_{\mathbf{G}}(x, y) &= \max(x, y) \\ \tilde{\rightarrow}_{\mathbf{G}}(x, y) &= \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}\end{aligned}$$

m -valued Gödel logic is defined the same, except $V = V_m$.

Proposition inf.2. *The logic \mathbf{G}_3 defined by ?? is the same as \mathbf{G}_3 defined by [Definition inf.1](#).*

Proof. This can be seen by comparing the truth tables for the connectives given in ?? with the truth tables determined by the equations in [Definition inf.1](#):

$\tilde{\neg}_{\mathbf{G}_3}$	1	0	0	$\tilde{\wedge}_{\mathbf{G}}$	1	1/2	0	□
	1	0	1		1	1/2	0	
	1/2	0	1/2		1/2	1/2	0	
	0	1	0		0	0	0	
$\tilde{\vee}_{\mathbf{G}}$	1	1/2	0	$\tilde{\rightarrow}_{\mathbf{G}}$	1	1/2	0	□
	1	1	1		1	1/2	0	
	1/2	1	1/2		1/2	1	0	
	0	1	1/2		0	1	1	

mvl:inf:god:
prop:god-infty-m

Proposition inf.3. *If $\Gamma \vDash_{\mathbf{G}_\infty} \psi$ then $\Gamma \vDash_{\mathbf{G}_m} \psi$ for all $m \geq 2$.*

Proof. Exercise. □

Problem inf.1. Prove **Proposition inf.3**.

In fact, the converse holds as well.

Like \mathbf{G}_3 , \mathbf{G}_∞ has all intuitionistically valid formulas as tautologies, and the same examples of non-tautologies are non-tautologies of \mathbf{G}_∞ :

$$\begin{array}{ll} p \vee \neg p & (p \rightarrow q) \rightarrow (\neg p \vee q) \\ \neg\neg p \rightarrow p & \neg(\neg p \wedge \neg q) \rightarrow (p \vee q) \\ ((p \rightarrow q) \rightarrow p) \rightarrow p & \neg(p \rightarrow q) \rightarrow (p \wedge \neg q) \end{array}$$

The example of an intuitionistically invalid formula that is nevertheless a tautology of \mathbf{G}_3 , $(p \rightarrow q) \vee (q \rightarrow p)$, is also a tautology in \mathbf{G}_∞ . In fact, \mathbf{G}_∞ can be characterized as intuitionistic logic to which the schema $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ is added. This was shown by Michael Dummett, and so \mathbf{G}_∞ is often referred to as Gödel–Dummett logic **LC**.

Problem inf.2. Show that $(p \rightarrow q) \vee (q \rightarrow p)$ is a tautology of \mathbf{G}_∞ .

Problem inf.3. Show that $(p \rightarrow q) \vee (q \rightarrow r) \vee (r \rightarrow s)$, which is a tautology of \mathbf{G}_3 , is not a tautology of \mathbf{G}_∞ .

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Bibliography