syn.1  Unique Readability

We may wonder if for each term there is a unique way of forming it, and there
is. For each lambda term there is only one way to construct and interpret it.
In the following discussion, a \textit{formation} is the procedure of constructing a term
using the formation rules (one or several times) of \textit{??}.

\textbf{Lemma syn.1.} \textit{A term starts with either a variable or a parenthesis.}

\textit{Proof.} Something counts as a term only if it is constructed according to \textit{??}. If
it is the result of \textit{??}, it must be a variable. If it is the result of \textit{??} or \textit{??}, it
starts with a parenthesis. \qed

\textbf{Lemma syn.2.} \textit{The result of an application starts with either two parentheses
or a parenthesis and a variable.}

\textit{Proof.} If \( M \) is the result of an application, it is of the form \((PQ)\), so it begins
with a parenthesis. Since \( P \) is a term, by \textbf{Lemma syn.1}, it begins either with
a parenthesis or a variable. \qed

\textbf{Lemma syn.3.} \textit{No proper initial part of a term is itself a term.}

\textbf{Problem syn.1.} Prove \textbf{Lemma syn.3} by induction on the length of terms.

\textbf{Proposition syn.4 (Unique Readability).} \textit{There is a unique formation
for each term. In other words, if a term \( M \) is formed by a formation, then it
is the only formation that can form this term.}

\textit{Proof.} We prove this by induction on the formation of terms.

1. \( M \) is of the form \( x \), where \( x \) is some variable. Since the results of abstrac-
tions and applications always start with parentheses, they cannot have
been used to construct \( M \); Thus, the formation of \( M \) must be a single
step of \textit{??}.

2. \( M \) is of the form \((\lambda x. N)\), where \( x \) is some variable and \( N \) is a term. It
could not have been constructed according to \textit{??}, because it is not a
single variable. It is not the result of an application, by \textbf{Lemma syn.2}. Thus \( M \) can only be the result of an abstraction on \( N \). By inductive
hypothesis we know that formation of \( N \) is itself unique.

3. \( M \) is of the form \((PQ)\), where \( P \) and \( Q \) are terms. Since it starts with
a parentheses, it cannot also be constructed by \textit{??}. By \textbf{Lemma syn.1},
\( P \) cannot begin with \( \lambda \), so \((PQ)\) cannot be the result of an abstraction.
Now suppose there were another way of constructing \( M \) by application,
e.g., it is also of the form \((P'Q')\). Then \( P \) is a proper initial segment
of \( P' \) (or vice versa), and this is impossible by \textbf{Lemma syn.3}. So \( P \) and
\( Q \) are uniquely determined, and by inductive hypothesis we know that
formations of \( P \) and \( Q \) is unique. \qed
A more readable paraphrase of the above proposition is as follows:

**Proposition syn.5.** A term $M$ can only be one of the following forms:

1. $x$, where $x$ is a variable uniquely determined by $M$.

2. $(\lambda x. N)$, where $x$ is a variable and $N$ is another term, both of which is uniquely determined by $M$.

3. $(PQ)$, where $P$ and $Q$ are two terms uniquely determined by $M$.

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Bibliography