syn.1 Unique Readability

lam:syn:unq:sec

We may wonder if for each term there is a unique way of forming it, and there is. For each lambda term there is only one way to construct and interpret it. In the following discussion, a *formation* is the procedure of constructing a term using the formation rules (one or several times) of **??**.

lam:syn:unq:lem:term-start

Lemma syn.1. *A term starts with either a variable or a parenthesis.*

Proof. Something counts as a term only if it is constructed according to **??**. If it is the result of **??**, it must be a variable. If it is the result of **??** or **??**, it starts with a parenthesis. □

lam:syn:unq:lem:app-start

Lemma syn.2. *The result of an application starts with either two parentheses or a parenthesis and a variable.*

Proof. If M is the result of an application, it is of the form (PQ), so it begins with a parenthesis. Since P is a term, by Lemma syn.1, it begins either with a parenthesis or a variable. □

lam:syn:unq:lem:initial

Lemma syn.3. *No proper initial part of a term is itself a term.*

Problem syn.1. Prove Lemma syn.3 by induction on the length of terms.

lam:syn:unq:prop:unq

Proposition syn.4 (Unique Readability). *There is a unique formation for each term. In other words, if a term M is formed by a formation, then it is the only formation that can form this term.*

Proof. We prove this by induction on the formation of terms.

1. M is of the form x, where x is some variable. Since the results of abstractions and applications always start with parentheses, they cannot have been used to construct M; Thus, the formation of M must be a single step of **????**.

2. M is of the form $(\lambda x.\, N)$, where x is some variable and N is a term. It could not have been constructed according to **????**, because it is not a single variable. It is not the result of an application, by Lemma syn.2. Thus M can only be the result of an abstraction on N. By inductive hypothesis we know that formation of N is itself unique.

3. M is of the form (PQ), where P and Q are terms. Since it starts with a parentheses, it cannot also be constructed by **????**. By Lemma syn.1, P cannot begin with λ, so (PQ) cannot be the result of an abstraction. Now suppose there were another way of constructing M by application, e.g., it is also of the form $(P'Q')$. Then P is a proper initial segment of P' (or vice versa), and this is impossible by Lemma syn.3. So P and Q are uniquely determined, and by inductive hypothesis we know that formations of P and Q is unique. □

A more readable paraphrase of the above proposition is as follows:

Proposition syn.5. *A term M can only be one of the following forms:*

1. *x, where x is a variable uniquely determined by M.*
2. *$(\lambda x. N)$, where x is a variable and N is another term, both of which is uniquely determined by M.*
3. *(PQ), where P and Q are two terms uniquely determined by M.*

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Bibliography