syn.1 Substitution

Free variables are references to environment variables, thus it makes sense to actually use a specific value in the place of a free variable. For example, we may want to replace $f$ in $\lambda x. fx$ with a specific term, like the identity function $\lambda y. y$. This results in $\lambda x. (\lambda y. y)x$. The process of replacing free variables with lambda terms is called substitution.

**Definition syn.1 (Substitution).** The substitution of a term $N$ for a variable $x$ in a term $M$, $M[N/x]$, is defined inductively by:

1. $x[N/x] = N$.
2. $y[N/x] = y$ if $x \neq y$.
3. $PQ[N/x] = (P[N/x])(Q[N/x])$.
4. $(\lambda y.P)[N/x] = \lambda y. P[N/x]$, if $x \neq y$ and $y \notin \text{FV}(N)$, otherwise undefined.

In Definition syn.1(4), we require $x \neq y$ because we don’t want to replace *bound* occurrences of the variable $x$ in $M$ by $N$. For example, if we compute the substitution $\lambda x. x[y/x]$, the result should not be $\lambda x. y$ but simply $\lambda x. x$.

When substituting $N$ for $x$ in $\lambda y. P$, we also require that $y \notin \text{FV}(N)$. For example, we cannot substitute $y$ for $x$ in $\lambda y. x$, i.e., $\lambda y. x[y/x]$, because it would result in $\lambda y. y$, a term that stands for the function that accepts an argument and returns it directly. But the term $\lambda y. x$ stands for a function that always returns the term $x$ (or whatever $x$ refers to). So the result we actually want is a function that accepts an argument, drop it, and returns the environment variable $y$. To do this properly, we would first have to “rename” the bound variable $y$.

**Problem syn.1.** What is the result of the following substitutions?

1. $\lambda y. x(\lambda w. uvx)[(uv)/x]$
2. $\lambda y. x(\lambda x. x)[(\lambda y. xy)/x]$
3. $y(\lambda v. xv)[(\lambda y. vy)/x]$

**Theorem syn.2.** If $x \notin \text{FV}(M)$, then $\text{FV}(M[N/x]) = \text{FV}(M)$, if the left-hand side is defined.

**Proof.** By induction on the formation of $M$.

1. $M$ is a variable: exercise.
2. $M$ is of the form $(PQ)$: exercise.
3. $M$ is of the form $\lambda y. P$, and since $\lambda y. P[N/x]$ is defined, it has to be $\lambda y. P[N/x]$. Then $P[N/x]$ has to be defined; also, $x \neq y$ and $x \notin \text{FV}(Q)$. Then:

$$\text{FV}(\lambda y. P[N/x]) =$$
$$= \text{FV}(\lambda y. P[N/x]) \quad \text{by (4)}$$
$$= \text{FV}(P[N/x]) \setminus \{y\} \quad \text{by ???}$$
$$= \text{FV}(P) \setminus \{y\} \quad \text{by inductive hypothesis}$$
$$= \text{FV}(\lambda y. P) \quad \text{by ???}\, \square$$

Problem syn.2. Complete the proof of Theorem syn.2.

Theorem syn.3. If $x \in \text{FV}(M)$, then $\text{FV}(M[N/x]) = (\text{FV}(M) \setminus \{x\}) \cup \text{FV}(N)$, provided the left hand is defined.

Proof. By induction on the formation of $M$.

1. $M$ is a variable: exercise.

2. $M$ is of the form $PQ$: Since $(PQ)[N/y]$ is defined, it has to be $(P[N/x])(Q[N/x])$ with both substitution defined. Also, since $x \in \text{FV}(PQ)$, either $x \in \text{FV}(P)$ or $x \in \text{FV}(Q)$ or both. The rest is left as an exercise.

3. $M$ is of the form $\lambda y. P$. Since $\lambda y. P[N/x]$ is defined, it has to be $\lambda y. P[N/x]$, with $P[N/x]$ defined, $x \neq y$ and $y \notin \text{FV}(N)$; also, since $y \in \text{FV}(\lambda x. P)$, we have $y \in \text{FV}(P)$ too. Now:

$$\text{FV}((\lambda y. P)[N/x]) =$$
$$= \text{FV}(\lambda y. P[N/x])$$
$$= \text{FV}(P[N/x]) \setminus \{y\}$$
$$= ((\text{FV}(P) \setminus \{y\}) \cup (\text{FV}(N) \setminus \{x\}) \quad \text{by inductive hypothesis} \, \square$$
$$= (\text{FV}(P) \setminus \{x, y\}) \cup \text{FV}(N) \quad x \notin \text{FV}(N)$$
$$= (\text{FV}(\lambda y. P) \setminus \{x\}) \cup \text{FV}(N)$$

Problem syn.3. Complete the proof of Theorem syn.3.

Theorem syn.4. $x \notin \text{FV}(M[N/x])$, if the right-hand side is defined and $x \notin \text{FV}(N)$.

Proof. Exercise. \, \square

Problem syn.4. Prove Theorem syn.4.
Theorem syn.5. If $M[y/x]$ is defined and $y \notin \text{FV}(M)$, then $M[y/x][x/y] = M$.

Proof. By induction on the formation of $M$.

1. $M$ is a variable $z$: Exercise.

2. $M$ is of the form $(PQ)$. Then:

   $$(PQ)[y/x][x/y] = ((P[y/x])(Q[y/x]))[x/y]$$
   $$= (P[y/x][x/y])(Q[y/x][x/y])$$
   $$= (PQ) \text{ by inductive hypothesis}$$

3. $M$ is of the form $\lambda z. N$. Because $\lambda z. N[y/x]$ is defined, we know that $z \neq y$. So:

   $$(\lambda z. N)[y/x][x/y]$$
   $$= (\lambda z. N[y/x])[x/y]$$
   $$= \lambda z. N[y/x][x/y]$$
   $$= \lambda z. N \text{ by inductive hypothesis}$$

Problem syn.5. Complete the proof of Theorem syn.5.

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Bibliography