There is another relation on λ terms. In ?? we used the example λx. (fx), which accepts an argument and applies f to it. In other words, it is the same function as f: λx. (fx)N and fN both reduce to fN. We use η-reduction (and η-extension) to capture this idea.

**Definition syn.1 (η-contraction, →η).** η-contraction (→η) is the smallest compatible relation on terms satisfying the following condition:

$$\lambda x. M x \rightarrow_{\eta} M \text{ provided } x \notin FV(M)$$

**Definition syn.2 (βη-reduction, →betaeta).** βη-reduction (→betaeta) is the smallest reflexive, transitive relation on terms containing →β and →η, i.e., the rules of reflexivity and transitive plus the following two rules:

1. If $M \rightarrow_{\beta} N$ then $M \rightarrow_{\beta\eta} N$.
2. If $M \rightarrow_{\eta} N$ then $M \rightarrow_{\beta\eta} N$.

**Definition syn.3.** We extend the equivalence relation = with the η-contraction rule:

$$\lambda x. fx = f$$

and denote the extended relation as ≅. η-equivalence is important because it is related to extensionality of lambda terms:

**Definition syn.4 (Extensionality).** We extend the equivalence relation = with the (ext) rule:

If $Mx = Nx$ then $M = N$, provided $x \notin FV(MN)$.

and denote the extended relation as ≅ext.

Roughly speaking, the rule states that two terms, viewed as functions, should be considered equal if they behave the same for the same argument.

We now prove that the η rule provides exactly the extensionality, nothing else.

**Theorem syn.5.** $M \equiv_{ext} N$ if and only if $M \equiv_{\eta} N$.

**Proof.** First we prove that $\equiv_{\eta}$ is closed under the extensionality rule. That is, ext rule doesn’t add anything to $\equiv_{\eta}$. We then have $\equiv_{\eta}$ contains $\equiv_{ext}$, and if $M \equiv_{ext} N$, then $M \equiv_{\eta} N$. 

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To prove $\eta$ is closed under $\text{ext}$, note that for any $M = N$ derived by the $\text{ext}$ rule, we have $Mx \equiv Nx$ as premise. Then we have $\lambda x. Mx \equiv \lambda x. Nx$ by a rule of $\equiv$, applying $\eta$ on both side gives us $M \equiv N$.

Similarly we prove that the $\eta$ rule is contained in $\equiv$. For any $\lambda x. Mx$ and $M$ with $x \notin \text{FV}(M)$, we have that $(\lambda x. Mx)x \equiv Mx$, giving us $\lambda x. Mx \equiv M$ by the $\text{ext}$ rule.

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Bibliography