There is another relation on λ terms. In ?? we used the example \( \lambda x. (fx) \), which accepts an argument and applies \( f \) to it. In other words, it is the same function as \( \lambda x. (fx)N \) and \( fN \) both reduce to \( fN \). We use \( \eta \)-reduction (and \( \eta \)-extension) to capture this idea.

Definition syn.1 (\( \eta \)-contraction, \( \eta \rightarrow \)). \( \eta \)-contraction \( (\eta \rightarrow) \) is the smallest compatible relation on terms satisfying the following condition:

\[ \lambda x. Mx \eta \rightarrow M \text{ provided } x \notin \text{FV}(M) \]

Definition syn.2 (\( \beta \eta \)-reduction, \( \beta \eta \rightarrow \)). \( \beta \eta \)-reduction \( (\beta \eta \rightarrow) \) is the smallest reflexive, transitive relation on terms containing \( \beta \rightarrow \) and \( \eta \rightarrow \), i.e., the rules of reflexivity and transitive plus the following two rules:

1. If \( M \beta \rightarrow N \) then \( M \beta \eta \rightarrow N \).
2. If \( M \eta \rightarrow N \) then \( M \beta \eta \rightarrow N \).

Definition syn.3. We extend the equivalence relation \( = \) with the \( \eta \)-conversion rule:

\[ \lambda x. fx = f \]

and denote the extended relation as \( \eta = \) .

\( \eta \)-equivalence is important because it is related to extensionality of lambda terms:

Definition syn.4 (Extensionality). We extend the equivalence relation \( = \) with the \( (\text{ext}) \) rule:

If \( Mx = Nx \) then \( M = N \), provided \( x \notin \text{FV}(MN) \).

and denote the extended relation as \( \equiv \).

Roughly speaking, the rule states that two terms, viewed as functions, should be considered equal if they behave the same for the same argument.

We now prove that the \( \eta \) rule provides exactly the extensionality, and nothing else.

Theorem syn.5. \( M \equiv N \) if and only if \( M \eta N \).

Proof. First we prove that \( \eta = \) is closed under the extensionality rule. That is, \( \text{ext} \) rule doesn’t add anything to \( \equiv \). We then have \( \equiv \) contains \( \equiv \), and if \( M \equiv N \), then \( M \eta N \).
To prove $\eta$ is closed under $ext$, note that for any $M = N$ derived by the $ext$ rule, we have $Mx \equiv Nx$ as premise. Then we have $\lambda x. Mx \equiv \lambda x. Nx$ by a rule of $=\$, applying $\eta$ on both side gives us $M \equiv N$.

Similarly we prove that the $\eta$ rule is contained in $\equiv$. For any $\lambda x. Mx$ and $M$ with $x \notin FV(M)$, we have that $(\lambda x. Mx)x \equiv Mx$, giving us $\lambda x. Mx \equiv M$ by the $ext$ rule.

\[\square\]

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Bibliography