α-Equivalence is very natural, as terms that are α-equivalent “mean the same.” In fact, it is possible to give a syntax for lambda terms which does not distinguish terms that can be α-converted to each other. The best known replaces variables by their De Bruijn index.

When we write λx. M, we explicitly state that x is the parameter of the function, so that we can use x in M to refer to this parameter. In the de Bruijn index, however, parameters have no name and reference to them in the function body is denoted by a number denoting the levels of abstraction between them. For example, consider the example of λx. λy. yx: the outer abstraction is on binds the variable x; the inner abstraction binds the variable is y; the sub-term yx lies in the scope of the inner abstraction: there is no abstraction between y and its abstract λy, but one abstract between x and its abstract λx. Thus we write 0 1 for yx, and λ. λ. 01 for the entire term.

**Definition syn.1.** De Bruijn terms are inductively defined as follows:

1. n, where n is any natural number.
2. PQ, where P and Q are both De Bruijn terms.
3. λ. N, where N is a De Bruijn term.

A formalized translation from ordinary lambda terms to De Bruijn indexed terms is as follows:

**Definition syn.2.**

\[ F_\Gamma(x) = \Gamma(x) \]
\[ F_\Gamma(PQ) = F_\Gamma(P)F_\Gamma(Q) \]
\[ F_\Gamma(\lambda x. N) = \lambda x. F_{\Gamma[x]}(N) \]

where \( \Gamma \) is a list of variables indexed from zero, and \( \Gamma(x) \) denotes the position of the variable x in \( \Gamma \). For example, if \( \Gamma \) is x, y, z, then \( \Gamma(x) \) is 0 and \( \Gamma(z) \) is 2.

\( x, \Gamma \) denotes the list resulted from pushing \( x \) to the head of \( \Gamma \); for instance, continuing the \( \Gamma \) in last example, \( w, \Gamma \) is \( w, x, y, z \).

Recovering a standard lambda term from a de Bruijn term is done as follows:

**Definition syn.3.**

\[ G_\Gamma(n) = \Gamma[n] \]
\[ G_\Gamma(PQ) = G_\Gamma(P)G_\Gamma(Q) \]
\[ G_\Gamma(\lambda . N) = \lambda x. G_{\Gamma[x]}(N) \]

where \( \Gamma \) is again a list of variables indexed from zero, and \( \Gamma[n] \) denotes the variable in position n. For example, if \( \Gamma \) is x, y, z, then \( \Gamma[1] \) is y.

The variable x in last equation is chosen to be any variable that not in \( \Gamma \).
Here we give some results without proving them:

**Proposition syn.4.** If $M \xrightarrow{α} M'$, and $Γ$ is any list containing $\text{FV}(M)$, then $F_Γ(M) \equiv F_Γ(M')$.

Photo Credits

Bibliography