int.1 β-reduction

lam:int:bet:sec When we see $(\lambda m.\,(\lambda y.\,y)m)$, it is natural to conjecture that it has some connection with $\lambda m.\,m$, namely the second term should be the result of "simplifying" the first. The notion of *β-reduction* captures this intuition formally.

lam:int:bet:defn:betacontr **Definition int.1 (β-contraction, $\xrightarrow{\beta}$).** *The β-contraction ($\xrightarrow{\beta}$) is the smallest compatible relation on terms satisfying the following condition:*

$$(\lambda x.\,N)Q \xrightarrow{\beta} N[Q/x]$$

We say P is *β-contracted* to Q if $P \xrightarrow{\beta} Q$. A term of the form $(\lambda x.\,N)Q$ is called a *redex*.

lam:int:bet:prob:def **Problem int.1.** Spell out the equivalent inductive definitions of β-contraction as we did for change of bound variable in **??**.

lam:int:bet:defn:betared **Definition int.2 (β-reduction, $\xrightarrow{\beta}\!\!\!\rightarrow$).** *$\beta$-reduction* ($\overset{\beta}{\twoheadrightarrow}$) is the smallest reflexive, transitive relation on terms containing $\xrightarrow{\beta}$. We say P is β-reduced to Q if $P \overset{\beta}{\twoheadrightarrow} Q$.

We will write $\rightarrow$ instead of $\xrightarrow{\beta}$, and $\twoheadrightarrow$ instead of $\overset{\beta}{\twoheadrightarrow}$ when context is clear.

Informally speaking, $M \overset{\beta}{\twoheadrightarrow} N$ if and only if M can be changed to N by zero or several steps of β-contraction.

Definition int.3 (β-normal). A term that cannot be β-contracted any further is said to be *β-normal*.

If $M \overset{\beta}{\twoheadrightarrow} N$ and N is β-normal, then we say N is a *normal form* of M. One may ask if the normal form of a term is unique, and the answer is yes, as we will see later.

Let us consider some examples.

1. We have

$$\begin{aligned}(\lambda x.\,xxy)\lambda z.\,z &\rightarrow (\lambda z.\,z)(\lambda z.\,z)y\\ &\rightarrow (\lambda z.\,z)y\\ &\rightarrow y\end{aligned}$$

2. "Simplifying" a term can actually make it more complex:

$$\begin{aligned}(\lambda x.\,xxy)(\lambda x.\,xxy) &\rightarrow (\lambda x.\,xxy)(\lambda x.\,xxy)y\\ &\rightarrow (\lambda x.\,xxy)(\lambda x.\,xxy)yy\\ &\rightarrow \ldots\end{aligned}$$

3. It can also leave a term unchanged:

$$(\lambda x.\, xx)(\lambda x.\, xx) \to (\lambda x.\, xx)(\lambda x.\, xx)$$

4. Also, some terms can be reduced in more than one way; for example,

$$(\lambda x.\, (\lambda y.\, yx)z)v \to (\lambda y.\, yv)z$$

by contracting the outermost application; and

$$(\lambda x.\, (\lambda y.\, yx)z)v \to (\lambda x.\, zx)v$$

by contracting the innermost one. Note, in this case, however, that both terms further reduce to the same term, zv.

The final outcome in the last example is not a coincidence, but rather illustrates a deep and important property of the lambda calculus, known as the Church-Rosser property.

digression In general, there is more than one way to β-reduce a term, thus many reduction strategies have been invented, among which the most common is the *natural strategy*. The natural strategy always contracts the *left-most* redex, where the position of a redex is defined as its starting point in the term. The natural strategy has the useful property that a term can be reduced to a normal form by some strategy iff it can be reduced to normal form using the natural strategy. In what follows we will use the natural stratuegy unless otherwise specified.

Definition int.4 (β-equivalence, $=$). *β-Equivalence* ($=$) is the relation inductively defined as follows:

1. $M = M$.
2. If $M = N$, then $N = M$.
3. If $M = N$, $N = O$, then $M = O$.
4. If $M = N$, then $PM = PN$.
5. If $M = N$, then $MQ = NQ$.
6. If $M = N$, then $\lambda x.\, M = \lambda x.\, N$.
7. $(\lambda x.\, N)Q = N[Q/x]$.

The first three rules make the relation an equivalence relation; the next three make it compatible; the last ensures that it contains β-contraction.

Informally speaking, two terms are β-equivalent if and only if one of them can be changed to the other in zero or more steps of β-contraction, or "inverse" of β-contraction. The inverse of β-contraction is defined so that M inverse-β-contracts to N iff N β-contracts to M.

Besides the above rules, we will extend the relation with more rules, and denote the extended equivalence relation as $\stackrel{X}{=}$, where X is the extending rule.

Photo Credits

Bibliography