When we see $(\lambda m. (\lambda y. y)m)$, it is natural to conjecture that it has some connection with $\lambda m. m$, namely the second term should be the result of “simplifying” the first. The notion of $\beta$-reduction captures this intuition formally.

**Definition int.1 ($\beta$-contraction, $\xrightarrow{\beta}$).** The $\beta$-contraction $(\xrightarrow{\beta})$ is the smallest compatible relation on terms satisfying the following condition:

$$(\lambda x. N)Q \xrightarrow{\beta} N[Q/x]$$

We say $P$ is $\beta$-contracted to $Q$ if $P \xrightarrow{\beta} Q$. A term of the form $(\lambda x. N)Q$ is called a redex.

**Problem int.1.** Spell out the equivalent inductive definitions of $\beta$-contraction as we did for change of bound variable in **?.

**Definition int.2 ($\beta$-reduction, $\xrightarrow{\beta\rightarrow}$).** $\beta$-reduction $(\xrightarrow{\beta\rightarrow})$ is the smallest reflexive, transitive relation on terms containing $\xrightarrow{\beta}$. We say $P$ is $\beta$-reduced to $Q$ if $P \xrightarrow{\beta\rightarrow} Q$.

We will write $\rightarrow$ instead of $\xrightarrow{\beta}$, and $\Rightarrow$ instead of $\xrightarrow{\beta\rightarrow}$ when context is clear.

Informally speaking, $M \xrightarrow{\beta} N$ if and only if $M$ can be changed to $N$ by zero or several steps of $\beta$-contraction.

**Definition int.3 ($\beta$-normal).** A term that cannot be $\beta$-contracted any further is said to be $\beta$-normal.

If $M \xrightarrow{\beta\rightarrow} N$ and $N$ is $\beta$-normal, then we say $N$ is a normal form of $M$. One may ask if the normal form of a term is unique, and the answer is yes, as we will see later.

Let us consider some examples.

1. We have

$$(\lambda x. xxy)\lambda z. z \rightarrow (\lambda z. z)(\lambda z. z)y \rightarrow (\lambda z. z)y \rightarrow y$$

2. “Simplifying” a term can actually make it more complex:

$$(\lambda x. xxy)(\lambda x. xxy) \rightarrow (\lambda x. xxy)(\lambda x. xxy)y \rightarrow (\lambda x. xxy)(\lambda x. xxy)yy \rightarrow \ldots$$
3. It can also leave a term unchanged:

$$(\lambda x. xx)(\lambda x. xx) \rightarrow (\lambda x. xx)(\lambda x. xx)$$

4. Also, some terms can be reduced in more than one way; for example,

$$(\lambda x. (\lambda y. yx)z)v \rightarrow (\lambda y. yv)z$$

by contracting the outermost application; and

$$(\lambda x. (\lambda y. yx)z)v \rightarrow (\lambda x. zx)v$$

by contracting the innermost one. Note, in this case, however, that both terms further reduce to the same term, $zv$.

The final outcome in the last example is not a coincidence, but rather illustrates a deep and important property of the lambda calculus, known as the Church-Rosser property.

digression

In general, there is more than one way to $\beta$-reduce a term, thus many reduction strategies have been invented, among which the most common is the natural strategy. The natural strategy always contracts the left-most redex, where the position of a redex is defined as its starting point in the term. The natural strategy has the useful property that a term can be reduced to a normal form by some strategy iff it can be reduced to normal form using the natural strategy. In what follows we will use the natural strategy unless otherwise specified.

**Definition int.4 ($\beta$-equivalence, $=$).** $\beta$-Equivalence ($=$) is the relation inductively defined as follows:

1. $M = M$.
2. If $M = N$, then $N = M$.
3. If $M = N$, $N = O$, then $M = O$.
4. If $M = N$, then $PM = PN$.
5. If $M = N$, then $MQ = NQ$.
6. If $M = N$, then $\lambda x. M = \lambda x. N$.
7. $(\lambda x. N)Q = N[Q/x]$.

The first three rules make the relation an equivalence relation; the next three make it compatible; the last ensures that it contains $\beta$-contraction.

Informally speaking, two terms are $\beta$-equivalent if and only if one of them can be changed to the other in zero or more steps of $\beta$-contraction, or “inverse” of $\beta$-contraction. The inverse of $\beta$-contraction is defined so that $M$ inverse-$\beta$-contracts to $N$ iff $N$ $\beta$-contracts to $M$. 
Besides the above rules, we will extend the relation with more rules, and denote the extended equivalence relation as $\overset{X}{\equiv}$, where $X$ is the extending rule.

Photo Credits

Bibliography