ldf.1 Truth Values and Relations

lam:ldf:tvr:sec

We can encode truth values in the pure lambda calculus as follows:

$$\text{true} \equiv \lambda x.\, \lambda y.\, x$$
$$\text{false} \equiv \lambda x.\, \lambda y.\, y$$

Truth values are represented as *selectors*, i.e., functions that accept two arguments and returning one of them. The truth value true selects its first argument, and false its second. For example, $\text{true}\, MN$ always reduces to M, while $\text{false}\, MN$ always reduces to N.

Definition ldf.1. We call a relation $R \subseteq \mathbb{N}^n$ λ-definable if there is a term R such that

$$R\,\overline{n_1}\ldots\overline{n_k} \xrightarrow{\beta}\!\!\!\!\rightarrow \text{true}$$

whenever $R(n_1, \ldots, n_k)$ and

$$R\,\overline{n_1}\ldots\overline{n_k} \xrightarrow{\beta}\!\!\!\!\rightarrow \text{false}$$

otherwise.

For instance, the relation IsZero $= \{0\}$ which holds of 0 and 0 only, is λ-definable by

$$\text{IsZero} \equiv \lambda n.\, n(\lambda x.\, \text{false})\,\text{true}.$$

How does it work? Since Church numerals are defined as iterators (functions which apply their first argument n times to the second), we set the initial value to be true, and for every step of iteration, we return false regardless of the result of the last iteration. This step will be applied to the initial value n times, and the result will be true if and only if the step is not applied at all, i.e., when $n = 0$.

On the basis of this representation of truth values, we can further define some truth functions. Here are two, the representations of negation and conjunction:

$$\text{Not} \equiv \lambda x.\, x\,\text{false}\,\text{true}$$
$$\text{And} \equiv \lambda x.\, \lambda y.\, xy\,\text{false}$$

The function "Not" accepts one argument, and returns true if the argument is false, and false if the argument is true. The function "And" accepts two truth values as arguments, and should return true iff both arguments are true. Truth values are represented as selectors (described above), so when x is a truth value and is applied to two arguments, the result will be the first argument if x is true and the second argument otherwise. Now And takes its two arguments x and y, and in return passes y and false to its first argument x. Assuming x is

a truth value, the result will evaluate to y if x is true, and to false if x is false, which is just what is desired.

Note that we assume here that only truth values are used as arguments to And. If it is passed other terms, the result (i.e., the normal form, if it exists) may well not be a truth value.

Problem ldf.1. Define the functions Or and Xor representing the truth functions of inclusive and exclusive disjunction using the encoding of truth values as λ-terms.

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Bibliography